INTRODUCTION

Section I. GEOMETRY AND GEOMETRY-TEACHING THROUGH THE AGES

Carmelo Mammana & Vinicio Villani

The main aim of the present ICMI study is to identify current major trends in geometry and to discuss implications and perspectives concerning its teaching, for today and for tomorrow. We shall be discussing various aspects, ranging from social and didactical issues to curriculum design and teacher preparation; but first, in order to achieve this purpose, it seems appropriate to have a brief look back, at the manifold and varying roles which geometry has played during the past (remote and recent). In fact, the various phases of the historical evolution of geometry are intimately related to the evolution of mankind itself.

1. The Growth of Geometry

Since prehistory, people have felt a strong desire to reproduce the multifaceted aspects of reality by stylized drawings, to decorate their artifacts by adding ornamental motifs formed by simple geometric shapes endowed with symmetries, and to shape their first constructions in regular geometric forms.

At this early stage, the *visual* aspect is thus predominant.

In a subsequent phase, the expansion of human communities leads to improvements in the overall structure and organization of their social life, and thus represents the starting point of early forms of civilization in several parts of the world: China, Mesopotamia, India, Egypt, Mexico, Greece. Now geometry meets mainly *utilitarian needs* for the measure of lengths, areas and volumes, or for tracing boundaries on the soil. Moreover, geometry plays a relevant instrumental role with respect to other sciences, such as architecture, geography and astronomy.

At this stage we can perceive a first attempt to rationalize (at least 'locally') the acquired geometric knowledge. Here *visual* and *computational* aspects are interwoven.

For historical and cultural reasons, the achievements of the Greek civilization have been the most influential ones for the development of geometry as a science. The interest in geometry shifts from practical needs towards

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a more abstract and 'global' rationalisation process, which culminates, around 300 B.C. in the systematization achieved by Euclid in his *Elements* and in the subsequent work of Apollonius, Archimedes, Ptolemy.

In this new phase, interest is mainly concentrated on the *conceptual* aspects of geometry.

The perfection of Euclid's treatise became a model and a prototype for a rational systematization of all fields of knowledge. For many centuries, geometry was praised as one of the most relevant disciplines for the cultural formation of scholars in liberal arts, from the mediaeval "Trivium/Quadrivium" until the Renaissance, and beyond. On the other hand, this same perfection of Euclid's treatise inhibited further progress within geometry itself, resulting in a 'freezing' of geometric knowledge for almost 2000 years within the Euclidean scheme.

It is therefore not surprising that, after many centuries of a general stagnation, original ideas in geometric research arose from stimuli coming from outside Euclidean geometry: during the fifteenth century from studies in an artistic milieu about perspective (Piero della Francesca, Leon Battista Alberti), in the seventeenth century from a melding of geometry and algebra (Descartes) and towards the end of the eighteenth century from a systematic study of the methods of representation of 3-dimensional objects through drawings, i.e. by descriptive geometry (Monge).

All these aspects (projective geometry, analytic geometry, descriptive geometry) were considered to be alien to the spirit of Euclidean geometry, and therefore did not interfere with the still unquestioned authority of Euclid's treatise. One must wait until the nineteenth century, in order to achieve an advance beyond Euclidean geometry, thanks to the discovery of non-Euclidean geometries (Gauss, Bolyai, Lobachevsky).

Just as the method of coordinates was intimately linked to the philosophical views of Descartes, so the discovery of non-Euclidean geometries was strictly interwoven with the philosophical debate about the sources of certainty of our knowledge (Kant).

The consciousness of the possibility of imagining alternatives to Euclidean geometry entailed, in a certain sense, a loss of the central role which had been attributed until then to Euclidean geometry within mathematics and within scientific knowledge in general.

On the other hand, non-Euclidean geometries contributed to stimulate a new era of research in the foundations of geometry (Riemann, Pasch, Peano, Veronese) which culminated in the *Erlangen Program* by Felix Klein (1872)[(1)] and in the publication of the *Grundlagen der Geometrie* by David Hilbert (1899). The revolutionary aspect of this new viewpoint is due to its high level of *abstraction*, with a consequent loss of the previous ties of geometry to perceptible reality.

[(1)] F. Klein. *Vergleichende Betrachtungen über neuere geometrische Forschungen.* Math. Ann. 43 (1893), 63-100. English Translation in N.Y. Math. Soc. Bull. 2 (1893), 215-249.

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But, as the perfection of Euclid's *Elements* had closed the phase of rationalisation of geometric knowledge in antiquity, so the perfection of Hilbert's *Grundlagen* caused a pause in modern research about the foundations of geometry.

In subsequent years, research into the *algebraic* aspects of the discipline acquired an increasingly important role, thanks also to the fact that in the meantime Dedekind, Cantor and Weierstrass had provided a rigorous foundation for the theory of real numbers. It is worthwhile to notice that, while until that time the 'certainties' of algebra were derived from supposed certainties in geometry, at the end of the nineteenth century the viewpoint is radically turned upside down: from now on it is algebra which provides firm 'models' for geometry. This allows us to envisage abstract structures of arbitrary high dimensions, with no direct link to sensory experience (remember the contributions of Riemann and Minkowski to differential geometry, and the influence of these achievements on Einstein's relativity theory).

At the same time, from the beginning of this century, new algebraic tools have been created for a general study of geometric objects, among which, most powerful and pervasive is the theory of *vector spaces.*

As a consequence, there is a further gain in generality, with a further distancing from geometric intuition.

More recently, it seems that we are experiencing a renewed interest in the visual aspects of geometry, but once more this research is growing mostly outside the milieu of the mathematical community. Just think about the achievements concerning *applications* of geometry in computer graphics, in image reconstruction and image processing, in pattern recognition and robotics. In all these research fields, the most active departments are those of computer science, engineering, chemistry and medicine.

Even without mentioning further aspects of advanced research, quoted e.g. by Dieudonné at ICME 4[(2)], and which may be linked to elementary geometry only in a mediated form, this short historical outline shows how geometry oscillated and still oscillates between *visual, computational, conceptual, algebraic, utilitarian* and *applicational aspects.*

2. School Curricula

All these shifts in research and applicational interests have direct and significant impact on the teaching of geometry at all school levels. Times of great appreciation and times of neglect alternate. As for school geometry, after the epoch of Klein and Hilbert, it seems that in most countries geometry teaching has entered a period of low tide. Even in those countries where geometry still maintains its former central position in school curricula, this fact seems to be due more to the persistence of tradition, than to a careful

(2) J. Dieudonné. *The Universal Domination of Geometry.* Zbl. Did. Math. 13 (1981), 5-7.

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analysis of the effect of a teaching of geometry in Euclidean style on the mathematical culture of contemporary pupils.

There is a general agreement that the diverse aspects of geometry mentioned above, although they originated in different eras and different cultural milieus, are all important even today. The point is that, when building a school curriculum, it is necessary to make choices, and choices depend on the main didactical aims which one would like to pursue. This need to choose is precisely what makes it so difficult to decide on a coherent and balanced geometry curriculum for the whole span of learning in schools, a curriculum which should be suitable, in terms of its contents, methods and motivations, for all pupils of a certain cultural and social environment. Moreover, in order to achieve a successful implementation of a given curriculum, several external constraints should also be taken into account (e.g. cultural traditions, the amount of time to be devoted to specific topics, textbooks, teacher preparation).

The curricular solutions, historically attempted, which have focussed on only one of the above-mentioned aspects of geometry while neglecting others, have not had a positive outcome. Specifically, it has been observed that it is impossible to bypass the earlier and more intuitive phases, and to limit the teaching of geometry to its formal and algebraic aspects. On the other hand, it is surely not right to ignore the formal aspects, which were once, and are still now, a model of rigorous deductive thinking. Nor is it right to ignore the algebraic aspects, which are, after all, the most powerful ones for further study at advanced level. It is therefore necessary to find appropriate points of equilibrium. But it is unlikely that a unique optimal solution, having universal validity, exists. In any case, the present ICMI study does not presume to propose such a solution. More modestly, our hope is that the consciousness of the complexity of the problems we are faced with, may help those who are charged with making choices, to take advantage of successful experiences, and to avoid errors already identified in previous attempts.

Section II. ABOUT THE PRESENT ICMI STUDY

Vinicio Villani

Due to the widespread concern about the role of geometry in school mathematics, at the closing session of the 7th International Congress on Mathematical Education in Quebec (1992), on behalf of the ICMI Executive Committee, President Miguel de Guzmán proposed the start of an ICMI Study on the teaching of geometry. He expressed the hope that the initiative might begin with a working conference which would take place in the Mediterranean region where geometry originated over 2000 years ago.

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In response, the Department of Mathematics of the University of Catania generously offered to host an international conference on this subject.

One year later, the Executive Committee of ICMI appointed an International Program Committee (IPC) consisting of Régine Douady, Vagn Lundsgaard Hansen, Rina Hershkowitz, Joseph Malkevitch, Carmelo Mammana (chair of the local organizing committee), Mogens Niss, Iman Osta, and Vinicio Villani (IPC chair).

Subsequent steps were:

• The development of a *Discussion Document* including a call for contributions, which was widely circulated during autumn 1994 (the Discussion Document is reprinted in the Appendix).

• The Catania Conference, by invitation, which was held from September 27 to October 2, 1995. There were 72 participants, from 33 countries.

The abstracts of the contributions to the conference were published in advance in a pre-proceedings volume, edited by Professor Mammana and distributed to all participants.

During the Conference some plenary lectures were given and two round tables were held, but the main part of the Conference was devoted to working groups, meeting in 6 parallel sessions.

A the end of the Catania Conference, the International Program Committee agreed upon the main guidelines of the present volume, and appointed coordinators for each chapter. Each coordinator in his turn involved other people, mainly, but not exclusively, from among participants to the Catania Conference.

A draft version of this volume was discussed and revised during a meeting of the program committee which took place at the International Centre for Mathematical Research (CIRM) in Trento during the week from January 15 to January 20, 1996.

The main guidelines and findings of the study were presented at the ICME-8 Congress in Sevilla (July 1996).

Of course, the content of the present study reflects mainly the views expressed in Catania, but it is not intended to be a collection of the individual contributions to that conference. Its aim is, instead, to highlight the main trends for an improvement of the teaching of geometry in the future, as they emerge from international research and actual school practice. We are well aware that there cannot be a unique prescription valid for all cultural, economical and social situations. Moreover we are aware that it is impossible to capture in one volume the richness of the extensive research work and practical experience of geometry educators from all over the world. And finally we are aware that every constructive change needs attention and hard work by mathematicians, educators, curriculum planners, textbook writers and teachers, in order to be put into practice.

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As for the structure of the book, Chapter 1 takes the reader immediately into the ongoing debate concerning the contrasting effects of traditional and innovative aspects of geometry and its applications.

Chapter 2 deals with the specific features of reasoning in geometry, including general aspects and specific case studies for different age groups, carried out in different countries.

Chapter 3 discusses the links between geometry and reality as well as the way geometry is being put to use in other branches of mathematics and knowledge.

Chapter 4 deals with the interplay of computer technology and geometry.

With Chapter 5 we are led to participate in what actually happens in the reality of a classroom.

Chapter 6 analyzes the evolution of geometry education since 1900 in several countries which at the beginning of this century had a leading role in geometrical research and teaching (England, Italy, France and Germany).

Chapter 7 concerns the impact of recent innovations on actual curricula, on the basis of five case studies from a sample of countries with quite different approaches to education (Colombia, Egypt, Poland, China and Germany).

Chapter 8 deals with assessment, and its influence on the teaching/learning process.

Chapter 9 discusses the vital role of an appropriate teacher education for the improvement of the teaching of geometry.

Chapter 10 aims to synthesize the most relevant findings of the study and to discuss possible actions to be undertaken in the near future, for an improvement of the teaching of geometry.

Each Chapter contains some specific references. Moreover, at the end of the volume a varied bibliography has been included, without any claim of completeness, where several significant classical and recent books about foundations, applications, teaching and learning of geometry are listed.

For reasons of textual economy, and not out of disrespect for female members, we shall use the third person pronoun "he" rather than "he/she" throughout the book to refer to both male and female teachers and pupils or students. Normally the word "pupil" will be used for the age group K (i.e. Kindergarten) to 18, while "student" will refer to learners in post-secondary education.

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As a concluding remark, we would like to stress the fact that, although the present study deals with geometry, and although we think that geometry should certainly be revitalized in mathematical teaching and learning, what the reader will find in this book should not be seen as an excessive attempt to lobby on behalf of geometry in competition with other topics or components of mathematics. We only claim that geometry should be assigned a balanced role in the teaching and learning of mathematics at large, in order to make mathematical education able to better pursue its general and overall purposes and goals.

Section III. ACKNOWLEDGEMENTS

Vinicio Villani

It is obvious from the description above that many people, institutions, and organizations have contributed in critical ways to the realization of this study.

In the first place, our thanks go to the members of the International Program Committee, the participants at the Catania Conference and the coordinators and contributors to the Chapters of this volume. Special appreciation goes to the organizers and hosts of the Catania Conference, in particular to Professor Carmelo Mammana, for the gracious hospitality given to those who attended.

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Unesco-Roste, chaired by Professor Angelo Marzollo, gave its sponsorship to the Study and provided financial support for travel expenses for several participants to the Catania Conference. Recipients of this support came from countries unable to provide resources for them to attend otherwise.

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As for the burden of the overall organization, I would like to express my warmest gratitude to the two colleagues who energetically and in timely manner assisted with proposals, provided constructive criticism and suggestions, and made valuable scientific contributions, from the beginning of the study until its final form: Professor Mogens Niss and Professor Carmelo Mammana.

The demanding and time-consuming task of producing a camera ready manuscript was carried out with great care by Dr. Jama Musse Jama (for the text) and by my son Marcello Villani (for the figures). My thanks to each of them.

Last but not least, the contributors to this study coming from non-English speaking countries are indebted to Professor Joseph Malkevitch and Professor Brian Griffiths for help with 'polishing' the English versions of their contributions.

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1. GEOMETRY: PAST AND FUTURE

Introduction

Since ancient times geometry has grown in both breadth and depth. Branches of geometry now exist which were not even conceived 75 years ago. This tremendous growth in geometry has created a tension for educators interested in the subject. There was no lack of topics competing for attention in the classroom at all levels 75 years ago. Now there are even more topic choices, new modes of teaching, and new technologies with which to deliver both new and old ideas alike.

In light of these tensions between the traditional and the new, Vagn Lundsgaard Hansen and Joseph Malkevitch were asked to frame the debate about these issues by giving a round-table devoted to the challenges created by the explosive growth in geometry for teachers. The two sections that follow are adaptations from the oral presentations made at the Catania Conference. They are followed in Section III by some thoughts of Adrien Douady, based on his reactions to the round-table.

Section I. EVERLASTING GEOMETRY

Vagn Lundsgaard Hansen

Mathematics has developed through millennia with its origin in the need for human beings to specify quantities and measure figures. The overwhelming strength that characterizes mathematics as a means to describe real-world problems lies in the interplay between the concrete and the abstract. It is a challenge to every teacher of mathematics to be aware of the dialectical symbiosis between the concrete and the abstract sides of mathematics.

In the school curriculum the handling of numbers is divided into the concrete: arithmetic, or computation with numbers, and the abstract: algebra, or computation with symbols. In the teaching of geometry this discrimination involves subtle things like distinguishing between a concrete figure and an abstract form and often remains hidden.

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I shall briefly go through some major developments in the history of geometry and indicate important milestones from the didactical point of view for the teaching of geometry. I shall point out some highlights in geometry, which in my opinion have everlasting importance and hence are relevant for the curriculum in geometry. Since time pressure is already quite strong in the educational system, it is difficult to introduce new topics in the curriculum without removing others. I shall argue that there are many classical topics which have a justified and important place in the curriculum. I hope, however, also to show how to enrich the study of traditional topics by pointing out some new aspects.

Computer graphics no doubt can enhance the teaching and understanding of most geometrical topics; you do not have to introduce new topics in order to make use of these new tools. In my opinion, old topics seen from a contemporary angle can be just as fresh and stimulating to pupils as new topics. And applications of the classical topics are numerous. There is a tendency to overlook this fact, maybe because the teaching of science has become rather descriptive and non-explanatory, i.e. non-mathematical, in many countries.

1. Euclidean geometry

Geometry derives from the greek word *geometria* ($\gamma\epsilon\omega\mu\epsilon\tau\rho\acute{\iota}\alpha$), which means measurement of land. The word was used by the greek historian Herodotus in the 5th century B.C. in his great epic on the Persian wars in which he writes that 'geometria' was used in the ancient Egypt to find the proper distribution of land after the annual floods of the Nile.

As a framework for the description and measurement of figures, geometry was developed empirically in many cultures several thousand years ago. Geometry as a science, which encompasses a collection of abstract statements about ideal forms and proofs of these statements, was founded around 600 B.C. in the Greek culture by Thales, who according to legend proved several theorems in geometry. The famous school of Pythagoras in the 6th century B.C. should also be mentioned in this connection. From that early period we shall, however, in particular single out Eudoxus (around 391–338 B.C.), who is known for a theory of proportions and the so-called method of exhaustion, which made possible rigorous determinations of areas and volumes.

Classical Greek geometry has first of all survived through the famous 13 books written by Euclid around 300 B.C. known as Euclid's *Elements*. In these books the mathematical, in particular the geometrical, knowledge possessed by the Greeks at the time of Euclid is summarized and systematized in such a way that the exposition has put a stamp on mathematical writings ever since.

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Teaching of *Euclidean geometry* is important from a very early stage in the school system. Children should be encouraged to study simple geometric figures and explore their properties. In early grades the approach to Euclidean geometry should mainly be informal and explanatory, leaving systematization to later grades. Morover, of course, even at later grades the teaching style should not strictly follow the style suggested by Euclid in the *Elements*. In many countries constructions with ruler and compass have disappeared from the syllabus, although this is a very good way of learning how to analyse a situation as the first step in a mathematical process. In the past it has also turned out to be a good way of creating an interest in mathematics among gifted children. Doing an elaborate construction is both creative and inventive. If you have to work out small programs on the computer for drawing geometrical figures you need to know these constructions. In fact, of these constructions, the important ones could again become central in the teaching of elementary geometry, using the computer as a tool.

Notions such as *similarity*, *congruence* and *symmetry* are fundamental for many mathematical arguments and applications of mathematics, and should be studied in some detail. At the advanced level such studies belong to *transformation geometry*.

I do not think children should formally be taught the postulates of Euclid, and certainly not at an early age, but their teachers should know them to teach with a proper perspective.

The abstract and concrete sides of geometry should not be formalized and made theoretical but should be experienced during the teaching and should be developed gradually in pupils and students. At the end the difference between a concrete *figure* and an abstract *form* should emerge. Proofs are useful when they act as explanations or reveal surprising facts, which you cannot establish by 'experimentation' alone. In my opinion one should always look for proofs which act as explanations, but I realize that sometimes it might be difficult. I also fully realize that what is a surprising fact for one child might not be surprising for another. But even so, I do think that there are facts which are surprising for almost anybody.

2. Conic sections

The second peak in classical Greek geometry was reached around 200 B.C. with the work on *conic sections* by Apollonius (c. 262–190 B.C.). Starting from purely mathematical interest, the conic sections have turned out to be useful in many very different contexts. It is of course of major importance that they enter into Kepler's description of planetary motions in the beginning of the 17th century; and later by Newton at the end of the 17th century when, in a major breakthrough in science, from his law of gravity he deduced the ellipse as the shape of orbits for planetary movements.

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But applications of conic sections are abundant. For example, the reflection properties of the ellipse are exploited in kidney-stone crushers, and those of the parabola in parabolic antennas. To realize certain mechanical movements of robots you need elliptical gear wheels. And the hyperbola is exploited in navigation (hyperbolic navigation, Decca Navigator systems). Without our thinking about it, the conic sections are part of our daily life in many ways.

The size of a particular ellipse is attached to the concrete figure. The abstract shape, which one might think of as the 'soul' of the figure, is for the ellipse characterized by *the eccentricity*, that measures how flattened it is.

Plato (427-348 B.C.) supposed that such shapes had an independent life in the world of ideas, which he assumed to be real. The great natural philosopher Aristotle (384-322 B.C.), who was the most important of Plato's pupils and immediate successors, does however distinguish between the real world and the world of ideas. The concrete figures live in the real world and the abstract shapes in the world of ideas. It is, among other things, due to this discrimination between the concrete and the abstract, that mathematicians at all later times have felt in great debt to the Greeks.

The teaching of conic sections should in my opinion stress the geometry of these objects as plane sections in conic surfaces, or – for the ellipse – in cylinders; cf. [4]. At first this approach may seem more difficult but there are many valuable pay-offs. In particular it helps to develop the spatial understanding. Computer graphics may be useful in familiarizing pupils with the conic sections, but it is more important than ever also to show real models of geometrical shapes you can touch and feel.

3. Analytic geometry

As is well known, the conic sections can be described by algebraic equations of second degree in two variables. It was in fact a main point for René Descartes (1596–1650) to obtain such an algebraisation of the conic sections in order to free their study from the geometrical arguments of Euclid and Apollonius, which he criticised for lacking a general method. He achieved his goal by the introduction of coordinate systems and the creation of *analytic geometry* (coordinate geometry), for which he laid the grounds in the book *La Géometrie* published in 1637. Independently, Pierre de Fermat (1601–1665) also developed a coordinate geometry. Although Fermat's discoveries go back to 1629, they were first published in 1679. Fermat, in contrast to Descartes, thought of analytic geometry just as an extension of the ideas of Euclid and Apollonius.

The methods which were developed to handle questions in geometry, by Euclid, Apollonius and their successors, prior to the development of analytic geometry, are now known by the name *synthetic geometry.*

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The methods of analytic geometry are of course of fundamental importance and belong to the curriculum in secondary schools in most countries. Unfortunately, I think that the focus on the algebraic description of e.g. the conic sections has made the treatment of them artificial and remote from applications, bringing them close to disappearing from the curriculum. This is an appalling mistake in my opinion.

The study of geometrical shapes described by algebraic equations in complete generality is an active area of research known as *algebraic geometry*. In this century algebraic geometry has developed enormously. Although algebraic geometry is at a high level of abstraction, it is also a basis for very deep and useful applications. As an example, methods from algebraic geometry are being used for the construction of unbreakable codes and for the construction of codes which in themselves can correct errors. In connection with electronic data transmission, such codes are already of far-reaching importance.

4. Non-Euclidean geometry

In Euclid's *Elements* there was one postulate, which sparked the curiosity of the mathematicians, namely *the parallel postulate*: "In the plane, given a line and a point outside this line, there passes exactly one line through the point, which does not intersect the given line." (This formulation is due to Playfair in 1795 and is the best known of many equivalent formulations.) Around 1830 the bomb exploded, when the Russian mathematician Nicolai Ivanovich Lobachevsky (1793–1856) in 1829 and the Hungarian mathematician János Bolyai (1802–1860) in 1832 independently publicized, that they could construct geometries, that satisfied all properties of Euclidean geometry except for the parallel postulate. This postulate thereby gained the status of an axiom that characterizes Euclidean geometry.(1)

The great German mathematician Carl Friedrich Gauss (1777–1855) had in fact obtained similar results already back in 1816 but had kept his findings to himself since they deviated so strongly from accepted philosophical thinking of the time that he feared to be ridiculed.

In a memoir of 1887, the French mathematician Henri Poincaré (1854–1912) described a concrete model of a non-Euclidean geometry in two dimensions, *the hyperbolic plane*; this model is now known as the Poincaré disc. The *points* in Poincaré's model of the hyperbolic plane are the points inside the boundary circle of a disc, and the *lines* are those circular arcs, which intersect the boundary circle orthogonally. One can equip the hyperbolic plane with a measure of length, such that certain distances are constant in the Euclidean geometry, but when measured by the *hyperbolic distance* they go to infinity when we approach the boundary circle. Angles are measured by their values as Euclidean angles.

(1) Notice that I here make a distinction between *postulate* and *axiom*, which might not be common in all countries.

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The birth of *non-Euclidean geometries* raised the question of which geometry describes the physical world in the best possible way. Thereby one of the golden periods in the interplay between mathematics and physics was initiated, which in the beginning of this century led to the development of Einstein's theory of *relativity*, cf. [5].

It is possible to present the construction of the hyperbolic plane at secondary school level, cf. [8]. The presentation contains many of the valuable constructions from classical Euclidean geometry taking as point of departure the construction of inversion in a circle. It is a very important episode in the history of geometry and gives a good opportunity for rethinking the role of Euclid's postulates. It contains the dramas and surprises you can hope for in a presentation of a piece of mathematics. Surely at least all teachers of mathematics should know about this.

It is easy to go on and study tilings of the hyperbolic plane and prove that the hyperbolic plane can be covered with identical regular hyperbolic n-gons for every integer $n \geq 3$. This is a surprising fact, in sharp contrast to Euclidean geometry, where you can only cover the Euclidean plane with identical regular n-gons for $n = 3, 4, 6$.

5. Optimality properties of geometrical objects

The study of optimality properties of geometrical objects is an important area which I think could add new dimensions to the teaching of geometry, maybe first of all at the secondary school level, but certain aspects can already be treated at the primary school level.

As an example, I shall mention *the isoperimetric problem*, cf. [2], [4], [6], [7]. In its simplest form the problem can be stated as follows: Find the closed plane curve (without self-intersections) of a fixed prescribed length that encloses the maximal plane area.

The ancient Greeks took it for granted that the solution to the problem is what they regarded as the most perfect of all closed curves, namely the circle. The Swiss mathematician Jacob Steiner (1796-1863) gave several ingenious arguments that the solution is the circle, but with the defect that he took it for granted that a solution exists. In [6], I have presented one of Steiner's arguments which I find particularly nice.

The problem was however first clarified completely in lectures at the University of Berlin in the 1870's by Karl Weierstrass (1815-1897) who pointed out that the difficult problem was in fact to prove the mere existence of a solution and developed methods to overcome this problem.

The general problem may be too difficult to treat. But the corresponding problem for quadrilaterals, where the optimal quadrilateral is the square, is rather easy, cf. [7]. The problem for triangles, where the optimal triangle is the equilateral triangle, is a little more difficult but can certainly be done at the secondary school level; cf. [4]. In connection with the discussion

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of triangles it would in this context also be quite natural to prove the seemingly rather deep theorem of Heron, expressing the area of a triangle by the lengths of its sides.

Optimality considerations enter into almost all physical and technological design problems. Optimal design of structures (girders, sheets, etc.) utilizing in the most efficient way the basic properties of materials, is a very important research area in modern technology, yet optimality questions are rarely raised in lower grades as mathematical questions.

6. Curved shapes

What is flat and what is curved? If you look through your mathematical glasses, you can easily be puzzled. The spiral staircases you find in various towers (the Round Tower in Copenhagen has a magnificent one) are obviously curved surfaces. But a spiral staircase can nevertheless be covered with straight boards radiating from the mid axis, as in winding stairs, where we imagine that the steps have been smoothed out. A surface, which in this way can be swept out by a line segment moving in 3-space, is called in mathematics a *ruled surface*. Ruled surfaces have many advantages from the point of view of constructions and hence they are used among other places in shipbuilding (for the design of the hull), and in buildings. The special type of ruled surface found in spiral staircases is called in mathematics a *helicoid*.

Helicoids have also another interesting property: on a smaller scale they can be produced as soap films. A surface in space which on small pieces follows the shape of a soap film is called a *minimal surface*. The name is due to the fact that minimal surfaces locally minimize area. They often occur in nature, e.g. in cell membranes, and they can be produced not only as soap films but also more permanently as fascinating membranes of glue. Helicoids are the only surfaces in 3-dimensional space which are at the same time ruled and minimal.

The study of minimal surfaces goes back to Euler (1707–1783), Lagrange (1736–1813), Meusnier (1754–1793) and others and is an active area of current research in mathematics. Around 1985 a new minimal surface in 3-space was found, which has attracted considerable interest. Computer graphics played a significant role in the first investigations of this surface and plays an important role in the continued studies of minimal surfaces.

Minimal surfaces satisfy nature's uncompromising demand for efficiency, and this makes them extra strong and stable. Since they are also aesthetically pleasing, they catch the interest of architects and engineers, as manifest in the works of the famous German architect Frei Otto.

Already at primary school level, it is certainly possible to produce objects which are actually models of minimal surfaces. At the secondary school level it is possible to go on and introduce curvature of surfaces after

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first having introduced curvature of plane curves by emphasizing the approximating circles of curvature; cf. [5]. This is probably only for the few but they also need challenges. It is curvature which makes life interesting!

7. From geometry to topology

In a paper from 1679, Leibniz (1646–1716) set out to formulate some fundamental geometrical properties of geometrical shapes, to use special symbols to represent them, and to combine these properties to create others. He called such studies *analysis situs*, or *geometria situs*. It is not quite clear what he meant, but in 1679, in a letter to Huygens he explained that he was not satisfied with analytic geometry to study geometrical figures, since it involved magnitudes. At first Leibniz did not stimulate any new developments with these ideas.

In 1735, Euler published a paper bearing the title *Solution of a problem from geometria situs* to solve the problem about the bridges in Königsberg. Rather than as a contribution to analysis situs, it is nowadays counted as the first proper result in *graph theory*. Then, in 1750, Euler published a proof of the theorem now known as *Euler's theorem on polyhedra*. This theorem is generally considered to be the first proper result in *analysis situs*; it only involves the combinatorial stucture of the surface of a convex polyhedron and not magnitudes.

The name *analysis situs* was commonly used for geometrical studies not directly involving magnitudes right up to the middle of this century although Listing, a student of Gauss, already in 1836 (in a letter written in Catania) had proposed to call it *topology*. The name topology is now generally attached to such studies of the qualitative properties of geometrical objects.

Topology has grown into a very important mathematical discipline in this century and it is therefore relevant to present elements of it at the secondary school level. It is rather easy both to present the Königberg bridge problem and Euler's theorem on polyhedra.

From this century, I think that it is possible to present the notion of mapping degree of maps of a circle into itself. This was introduced in 1910 by Brouwer (1881–1967) and has applications to fixed point theory, vector fields on spheres and the Borsuk antipode theorem; cf. [3].

It is also possible to present elements of knot theory and of discrete dynamical systems. In the latter case there are extensive possibilities for computer studies. Such new subjects should not be seen as alternatives to classical Euclidean geometry in two and three dimensions, but can be inserted into a geometry curriculum if time permits.

From the didactical point of view, I most of all like the modern developments which can naturally be linked to more classical topics to show the eternal values of our subject.

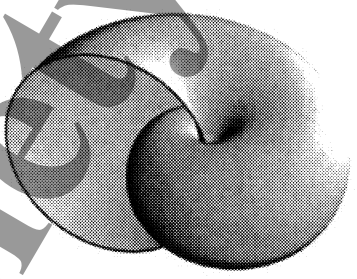
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8. The great book of geometry

In the living nature around us, the book of geometry lies open right before our eyes. Just think of the enchanting patterns in the wings of a butterfly, the fascinating symmetries in plants, and the fantastic shell constructions one finds among snails and mussels.

As an example, the curve of the shell of a snail is a *logarithmic spiral.* An especially perfect logarithmic spiral in nature is found in the shell of a primitive cuttlefish called the *Nautilus.* In the snail, the logarithmic spiral is a peaceful expression of *exponential growth.* Various kinds of growth are studied in other connections at the secondary school level and bringing this up in a geometrical context too underlines the universality of mathematics.

The shape of a logarithmic spiral is preserved by an arbitrary scaling up or down. Because of the high degree of self-similarity, the logarithmic spiral is often found in pictures of fractals. It is also the self-similarity which makes it possible that one can thicken the logarithmic spiral to a sausage with exponentially growing thickness so that the sausage can be wound up into a solid with the outside of one layer fitting exactly up against the inside of the next. The surface of the sausage is exactly the shell of a Nautilus, cf. [5]. If one understands the construction one can produce a very lifelike Nautilus on a computer. But nothing can be a substitute for the real world – whatever that is – not even a computer.



Computer study of a Nautilus by Steen Schyum Markvorsen
Technical University of Denmark

Mathematics has permanent residence in the shapes showing up in the many wonders of nature. If you see the underlying mathematical structure, you have access to the powerful, universal methods in mathematics, which are tied together by the overview gained by abstraction. For the strength of mathematics is to be found in the interplay between the concrete and the abstract.

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Section II. FINDING ROOM IN THE CURRICULUM FOR RECENT GEOMETRY

Joseph Malkevitch

Euclid would be very excited to be alive today! He would be astonished and pleased by how the subject he had done so much to develop and codify had prospered. He would probably be disappointed that he had not discovered such wonderful geometrical results as Euler's traversability theory for graphs and Euler's polyhedral formula. However, in light of so much geometry that had come after him, he might be surprised by the extent to which work that he did about 2500 years ago dominated the geometry curriculum, and still does.

Geometry, like the rest of mathematics and knowledge in general, has grown tremendously in the last 100 years. Although most of the growth in mathematical knowledge has few implications for pre-college curriculum (other than mentioning new applications for the benefit of society that new mathematics makes possible), this is not true for geometry. Geometry by its very nature has played a unique role within mathematics in the lower grades. This is precisely because geometry is less symbol-driven and hierarchical than other parts of mathematics. A surprisingly large part of recent work in geometry meets David Hilbert's test for a good mathematics

problem: it can be explained to the first person one walks up to on the street. Here are two examples.

1. The Art Museum Problem

Given a simple plane polygon P, which you can think of as a floor plan for a museum, determine the minimum number of guards (point surveillance devices) necessary to see the whole boundary of the polygon. A guard G can see any boundary point B, which lies on a straight line l from G to B, provided l does not contain points not in P.

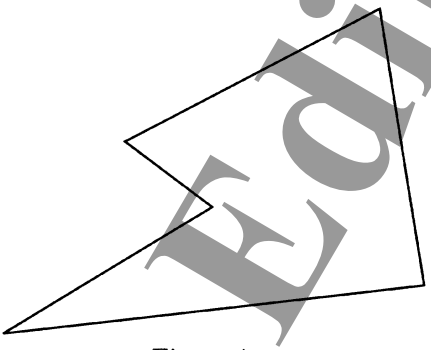


Figure 1 a

What is the minimum number of guards necessary to see the whole museum?

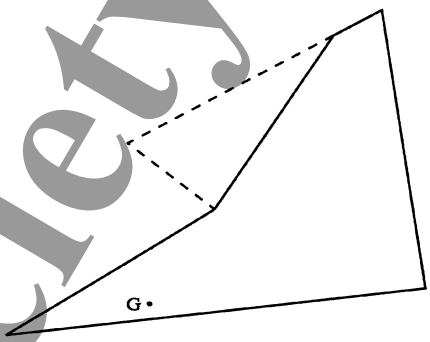


Figure 1 b

The portion of the museum visible by guard G is shaded.

2. The Delivery Problem

Figure 2 shows a portion of a town where a boy or girl must deliver advertising flyers to the homes that are located along the sections of street that are shown with stars (i.e. *****) in the figure. The boy or girl must start at his/her home, indicated by H in the map, where the flyers are dropped off by the distributor. After delivering the flyers to the required sections of

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the town, the deliverer must return home. If all the blocks have the same length, what is the shortest route possible?

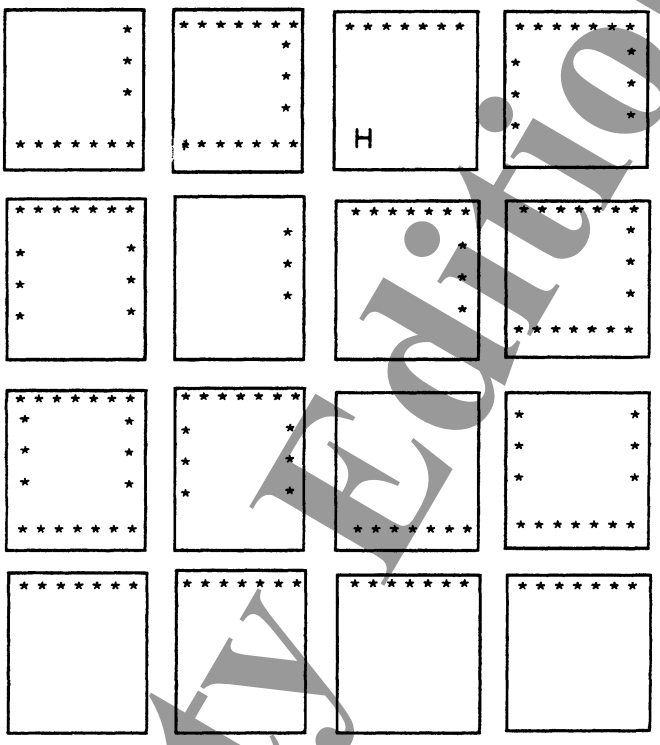


Figure 2

Find the shortest route starting and ending at H which visits all the starred edges.

Although both of these problems lead to very rich theories for mathematicians, they can be solved in specific cases like those shown in our previous examples by trial and error methods by anyone intrigued enough by their statements to work on them. Furthermore, although these problems could have been stated without any relation to an application context, they have been purposely phrased in a manner which suggests applications to the real world. For example, the second problem has significant applications to operations research work involving the delivery of mail, collecting garbage, and removing snow from roads.

These two examples may seem different from what many people think of as geometry. The geometry of Euclid, a remarkable accomplishment with both theoretical and practical implications, usually is developed in a deductive manner. It was almost surely a mathematical attempt to describe the physical world of the area and perimeter of rectangular fields and circular orbits, as well as related problems involving "space" in the local and global

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sense. The intellectual appeal of Euclid's *Elements* as a logically developed deductive system and the success of the facts of Euclidean geometry in astronomy, geography, trades (carpentry, machine shop, etc.) and in everyday life has led to the fact that this book has existed in more editions than any other book except the Bible!

The invention of non-Euclidean geometries and the importance of the parallel postulate have already been noted in Section I, 4.

Nowadays, however, almost no modern geometers are professionally interested in either the axiomatic development of geometry or in new theorems about various exotic points in circles, triangles, and quadrilaterals. But the legacy of 19th century geometry has been that these issues pervade geometry as taught in our schools. Nevetheless a dramatic transformation has occurred in the operational definition of geometry. For most practicing geometers, geometry has become the study of insights one is led to by mathematical training when one studies visual phenomena. Thus, a modern geometer will study knots, tilings, how to unblur a blurry photograph, etc. The range of what can be studied is as varied as what the visual world has to offer.

The unexpected surprise is that geometry, in the broad sense mentioned above, has led to a broad range of rich applications at the cutting edge of modern technology. It is perhaps not surprising that geometry has resulted in many applications in computer graphics, medical imaging and image processing, but what is perhaps more surprising is that dramatic work in communications technology and robotics has drawn on recent work in geometry.

Two examples will serve as illustrations: using geometrical thinking one can design codes that can be used to compress data. This data might take the form of text, pictures, television, or sound. Thus, fax technology is made possible not only by new achievements in engineering, but also by the codes that are used in a standard way as part of the fax technology. Similarly, one can design codes that can detect and/or correct errors using geometrical ideas. Innumerable new business and commercial applications are being found for such codes. Examples include sending X-rays between hospitals over phone hook-ups, tracking luggage better from start city to destination city (using bar-code technology), getting pictures back from spacecraft sent to other planets, storing sounds on a compact disc, etc.

These new developments create a quandary for geometry educators. Whenever new ideas arise that arguably deserve a part in the curriculum, a decision has to be made concerning what to do. This decision could assume one of the following three forms:

1. Though the new material could be added to the curriculum, a decision is made not to do so.

2. The new material is added by shortening the time devoted to the existing curriculum.

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3. Parts of the existing curriculum are removed in order to develop the new ideas.

I recommend that the third approach be adopted. Here are the reasons. Loosely speaking, the current curriculum (I write from an American perspective) with regard to geometry is structured in the following fashion. My description is obviously vastly over-simplified. In grades K-8 Euclidean geometry is taught in a manner such that pupils are taught:

1. The names of important shapes (e.g. circle, triangle, square, cube, cylinder, sphere, etc.) and practice locating these shapes in their everyday life (e.g. doors and windows are usually rectangles; manhole covers and coins are usually circles).

2. Developing basic concepts of distance between points, initially by measuring them with a ruler or walked strides. Eventually, the triangle inequality is discussed, and the fact that in right triangles the Pythagorean theorem holds.

3. The distinctions between the perimeter of a shape, its area, and its volume are developed and the results are derived for finding the perimeter, area, and volume of various shapes. (Learning these ideas requires making the connection between the use of numbers and the measurement of various quantities associated with shapes.)

4. Concepts of symmetry: point symmetry, rotations, reflections, and translations are described. Connections are often made with art and design. Sometimes polygons, polyhedra, and tilings of the plane are discussed and related to symmetry. Connections with the concept of a function are sometimes shown and roots of a "theory" of geometric transformations are developed.

5. Basic properties of point and line figures in the Euclidean plane (grades 6-8); sum of the angles of a triangle; angles associated with a line cutting a pair of parallel lines; vertical angles, etc.

6. Relating issues of size and proportion to shape. (If one doubles the radius of a sphere, what happens to the surface-area and volume of the sphere?)

In the grades 9-12 additional geometry is taught. Most of this is concentrated in the 10th grade, where historically a deductive treatment of Euclidean geometry was given. In recent years, educators have moved away from this approach to one in which students are encouraged to develop conjectures about Euclidean geometry, using such software systems as the Geometric Supposer, Geometer's Sketchpad or Cabri. After a conjecture is made and tested as well as it is possible using the computer environment, attempts are made to prove the conjecture based on earlier results. It is important to note that this route rarely takes the formal approach, emphasizing undefined terms and axioms of former years.

However, in the United States a dramatic drop-off in the systematic study of mathematics occurs starting in the 9th grade. A study has indicated that

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the number of students taking mathematics in high school each year after the 9th grade goes down by 1/2! Thus, only half the students who study algebra in the 9th grade go on to study the geometry described above, which is typically taught in the 10th grade.

The problem with the limited look at geometry in pre-college mathematics (as taught in America) is that it fails to deal with a variety of social and mathematical realities:

• The legal expectation that all Americans would graduate from high school is becoming closer to a reality. For example, in 1900 the percentage of 17-year-olds who graduated from high school was 6.4 percent. Now, this number is approximately 75 percent! Furthermore, more and more students are seeking higher education, in college.

• Professional geometers are rarely interested in axiomatics any more, and very few geometers are interested in the types of conjectures being explored and discovered using the software tools described above.

Given the changing needs of high school graduates and the nature of modern geometry, what is the alternative to what is now taught? Due to the very special nature of geometry, it is possible to teach geometry in the lower grades in a manner that:

– Connects up what is learned towards the end of the pupil's experience with geometry, to what is actually of interest to professionals who study geometry.

– Shows the emerging powerful new applications of geometry that have increasingly become part of every pupil's life (e.g. fax technology, CAT scans, compact disc technology, etc.).

– Retains the general goal for having geometry be part of a general curriculum (i.e. teach logical thinking, develop the concept of proof as being a tool unique to mathematics, provide useful mathematical tools for everyday life, provide a reasonable knowledge base for future scientists and mathematicians, etc.).

– Shows pupils the relation between geometry and the physics of space.

This new approach to the geometry curriculum can be based in part on the use of manipulatives and physical models (e.g. stick-and-ball and membrane models for polyhedra, tiles, and visual images) and on the use of computer software (e.g. LOGO). Such a curriculum emphasizes the visual aspects of geometry. It stresses discrete, combinatorial, non-metric, topological aspects of geometry as well as the applications of geometry. More details of these ideas can be found in [6] and briefly below.

What price do we pay for making a change of this sort? In my opinion almost none. Students who are identified as having a mathematical gift can be steered to one of the many books which treat the deductive theory of Euclidean geometry in more detail than the revised curriculum I would like to see. Since such a tiny part of the high school student population pursues careers in the mathematical sciences (e.g. mathematics, statistics,

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operations research, etc.) the gains for society are great: a public which is better educated about the nature of geometry and mathematics, and which can see – in a way they genuinely understand – the relation between doing mathematics for its own sake and the development of new technologies.

What are some of the new topics activities, and ideas that can be explored in pre-college classes of a geometrical kind? Here is a sample:

a. Graph theory

b. Compression codes and error-correcting codes.

c. Surfaces.

d. Frieze patterns, wallpaper patterns, fabric patterns.

e. Knots.

f. Polyhedra and tilings.

A natural question to ask is exactly what topics should be added to the current curriculum and which ones should be dropped from the current curriculum. In the elementary schools I believe that since so little is currently done, only additions should take place. Among the topics that deserve treatment (or more treatment) are knots, graph theory, tilings and polyhedra, and applications. However, in higher grades I believe that instead of devoting as much time to deductive geometry as is currently done, we should use only a sample of deductive geometry in high school, and treat deductive geometry as only one of the many arenas of geometric knowledge, methods, and ideas.

In conclusion, the traditional content of geometry in the later grades of high school is no less important for mathematics and science than it has ever been, but insistence on mastery of traditional 10th grade curriculum has not only continued the half-rate attrition mentioned above, but also contributed to negative stereotypes about mathematics (see [8]). New geometry topics (and their applications) should be added to the curriculum and some traditional material should be dropped from what we currently teach. One way to accomplish this would be to treat a sample of traditional topics from Euclid as typical of what happens in other parts of his work.

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Section III. SPACE AND PLANE

Adrien Douady

1. MASTERING THE RELATION WITH SPACE

Starting at age 3-4, children have to learn to locate themselves with respect to the surroundings. Also to locate others and objects with respect to themselves, or directly with respect to the surroundings. Later, in primary school, they make representations, which involve the transfer from 3D to 2D, and simultaneously from *mesospace* – space in which one lives – to *microspace* – space of small objects that one can tackle and move (cf. the contribution of R. Berthelot & M. H. Salin, in Chapter 2, Sect. V). They make descriptions of solid objects using words or drawings, and start using measures to make these descriptions more precise.

Building mental images of 3D configurations and anticipating a movement can be learned, starting very early. Work on this learning starts at primary school, and the mastering of complex situations requires several years.

The ability to make 2D representations of 3D configurations relies on the above mentioned competence and on some knowledge in 2D geometry. The ability to read 2D drawings, plans or charts, representing 3D configurations, to reason in 3D using these 2D representations requires all the above at a certain level. Naturally, this competence is acquired in a dialectic spiral.

Note that some sports (such as rock-climbing) or arts (many musical instruments) require a mastering of the relation with space which is essentially instinctive – at least not expressed by numbers or figures – but often needs to be extremely accurate. A steam-shovel driver usually has no degree in geometry, however he is able to produce very precise movements of his shovel via the numerous levers of his machine.

2. Technology in 3D

The extension of this process to technology is crucial in many fields: mechanics, automation, robotics, architecture, ship construction, etc. It involves giving a precise description of the position of a solid body in space. This is always a difficult problem, because there is no satisfactory system of coordinates for the set of all possible positions of a given solid, or equivalently the group of direct isometries of $\mathbb{R}^3$. If you look at positions up to translations, or equivalently, if you restrict to positions with a fixed point taken as origin, i.e. if you consider the orthogonal group in 3D, you can use the Euler angles, or the Bryant angles, or coordinates given by the axis and angle, or expression by quaternions or just by matrices. In these coordinate systems, the composition law is never simple. Still the quaternions are technically the most efficient.

This is what makes 3D geometry intrinsically more difficult than 2D geometry (for plane rotations around the origin, the composition is just given by the addition of angles).

When you are working on a problem in geometry, you are not necessarily dealing explicitly with the group of isometries of the space where you work, but as soon as the problem is intricate, difficulties will arise that one can analyse in terms of properties of this group. So I am tempted to say that this group is always present.

3. The teaching of planar geometry

Most of the teaching of geometry in secondary school deals with planar Euclidean geometry. Why is it so?

Geometry can be regarded as a very well modelled branch of physics, so that all the work is done in the model, described by axioms, by means of pure deductive reasoning. In this respect, 2D geometry is only a preliminary study that has to be mastered before facing the difficulties of 3D geometry.

But it is clear that this is not the main source of interest for planar geometry. Mostly, throughout the ages since the Greeks, Euclidean plane geometry has been studied by scholars and taught for its own sake, as a privileged place for learning and exercising deductive reasoning. The possibility of drawing figures helps the intuition, and often communication. But the game is then to give definitions and notations and make proofs according to strict rules, which can be understood without the help of figures.

The interplay between the mathematical language and the language of pictures, between the synthetic approach (where at each step what you say has a meaning in terms of the figure) and the analytic approach (using coordinates to transfer the questions to a numeric or algebraic framework, which allows blind calculation), between 2D and 3D, makes the subject extremely rich. A conic for instance can be defined as the intersection of a

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cone by a plane, or as a curve defined by an equation of degree 2, or by one of the various definitions in pure planar geometry. An essential part of a complete understanding of conics is to understand the equivalence between all these definitions, to have them all available at each moment, to be able to choose one or the other at convenience, and to transfer properties from one framework to another.

4. Computer versus Ruler and Compass

Recently, the possibility arose to draw figures by computer – even with a pocket calculator – with a better precision than with a ruler and a compass. Some softwares, such as Cabri, allow one to make constructions according to the same scheme as the classical constructions with ruler and compass.

One can also move the data (the arbitrary points the construction often starts with) and let the construction follow, or ask the computer whether 3 lines which apparently pass through a common point actually do so.

To master such a powerful tool writing sub-programs and using them as building blocks to avoid tedious repetitions (for example: circle passing through three points) is certainly as great a challenge as performing sophisticated constructions with ruler and compass starting from a few elementary ones (straight line passing through two points, intersections of two circles, etc).

Constructions with ruler and compass are pretty restricted. In algebraic terms, they allow us to reach only points whose coordinates belong to an iterated quadratic extension of the field generated by data. This restriction, imposed by the Greeks, is totally arbitrary. The computer can ignore it: in its view, a cubic or a logarithmic spiral is just as 'real' as a circle. However the software in use may have limitations (and one may impose some further ones, if it is regarded as desirable).

The real danger is that this jump in freedom might kill creativity. That is what happened to the architects all over the world in the decades which followed the introduction of concrete.

5. Non-Euclidean geometry

It was not a doubt about the adequacy of Euclidean geometry for a description of the physical world which led the ancient Greeks, and the Arabs in the middle ages, to question whether Euclid's postulate on parallels could be deduced from the other axioms of planar geometry. Indeed, the physical world is Euclidean with an extreme accuracy: the first discrepancy was observed during a solar eclipse as a confirmation of general relativity theory. And, if it were not precisely in view of this theory, the deviation observed could have been interpreted in many other ways.

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Hyperbolic geometry in 2D and also 3D, is an important cultural object because, by its existence, it proves the independence of the parallel postulate from the other axioms of Euclid, and so casts light on the foundations of geometry. It is also an extremely powerful tool in some specialized subjects (complex analysis, topology of manifolds). It may be accessible to some students at the end of secondary education who would be particularly motivated. But putting it in the curriculum, to be studied in any detail, would probably be unrealistic in view of the many constraints on the organization of teaching, and of the epistemological nature of the question which motivates only a minority.

Yet another type of geometry is that on the sphere. This raises no metaphysical problems since it can be realized in Euclidean 3D space. It gives a reasonably good model of the Earth, so it is culturally important. I do not know in how many countries it is now taught in secondary school. This requires at least a good level in 3D Euclidean geometry and in trigonometry. Geometry on the sphere differs from Euclidean geometry by the fact that there are several geodesics which join two antipodal points. Except for that, it is technically very similar to hyperbolic geometry.

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2. REASONING IN GEOMETRY

Section I. ABOUT REASONING IN GEOMETRY

Rina Hershkowitz

To Orly

Orly's doctoral research was devoted to studying the processes of justification and proving in geometry, used by the pupils she taught in the 9th and 10th grade. After two years of research her findings lead her in a new direction. The following is what she wrote in her research diary (a few weeks later Orly was killed in a car accident):

....After two years of searching for the difficulties students have in proving in geometry, I have started to wonder about the disadvantages of the classical way of teaching Euclidean geometry and have begun to experiment with other strategies. My intention is to create situations in which 'convincing' is necessary. In the class I stress the need to convince rather than the need to prove. In this way, kinds of reasoning that were never heard in this class in the past are legitimized. ...I create situations in which students themselves evaluate the 'level of conviction' of the justifications. This is possible when the class community is divided into two or more groups that have made different, or even opposing, conjectures.

In this situation, every argument passes a 'real test' – either it causes students from the opposing group to change their arguments, or not. I find that the successful ways of convincing are those that are based on deductive reasoning (including refutation by counterexample).

In the above, the main functions of reasoning are considered to be to understand, to explain and to convince. As a teacher and researcher, Orly (in the footsteps of others, e.g. Lampert [8]), gave her class the freedom to generate conjectures experimentally. When a conflict arose and the class split into two or more groups, debating different conjectures within the same situation, the pupils had the freedom to choose various kinds of arguments in order to try to convince each other. The arguments were judged by their power of conviction. This pedagogical approach reflects a considerable ongoing change in the mathematics education community, towards reasoning in mathematics as a whole, towards reasoning in geometry, and, in particular, towards proving.

It takes place within the framework of global changes of approach towards

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the question of what is meaningful learning in mathematics, and the question of what is considered by the mathematical community a mathematical proof and a proper mathematical argument (Hanna, [5]). Even at the risk of being simplistic, we suggest that the approach towards learning as a mere receptive process of knowledge transfer, and the view of mathematical argumentation as a very formal communication, have encouraged the classroom practice in which proving was a ritual governed by fixed rules, like the "two column proof"(1). Many other kinds of reasoning, such as inductive and/or visual reasoning, were permitted to be used in the mathematical activity, but were not considered as legitimate in the main mathematical product – the proof.

In contrast, new approaches towards learning, like the constructivist approach (Von Glaserfeld, [9]) and the socio-cultural approach (Vygotsky, [10]), which are having considerable impact, legitimize and encourage reasoning processes in the broadest sense. The learners, their understanding, and the interactions within the classroom community, are central to the view of the teaching-learning process. As a consequence, reasoning processes are now considered as a variety of actions that pupils take in order to communicate with, and explain to others, as well as to themselves, what they see, what they discover, and what they think and conclude.

The authors of the next five sections of this chapter discuss and illustrate examples of main new trends in reasoning in geometry.

In section II, Duval presents a meta-cognitive analysis of geometrical reasoning processes and their interactions with other thinking processes. Duval's contribution is a deep and interesting analysis, which regards reasoning as a holistic process in which 'proving' is only one of its three functions. The other two are: 'extension of knowledge' and 'explanation'. He also analyzes the interactions between the process of *reasoning* and other two thinking processes in geometry – *visualization* and *construction.*

The next two sections – section III by Bartolini Bussi & Boero, and section IV by Lehrer & Romberg – discuss theory, curriculum development and research which demonstrate what is called now "learning geometry from context". This trend will be discussed later in detail.

Berthelot & Salin (section V) discuss the roles of pupils' spatial knowledge in geometry learning. This contribution together with Duval's analysis of visualization and its interactions with reasoning in geometry, raise the issue of visual reasoning in geometry. This will be discussed in the last part of this contribution.

Jones (section VI) challenges the hierarchical approach, according to which intuitive reasoning should necessarily precede formal reasoning. His

(1) A "two column proof" refers to the process of giving 'local' reasons in one column, and references to agreed propositions in the other.

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example provides evidence that shows how pupils interweave intuitive-visual reasoning and deductive reasoning, while solving geometrical problems in a learning environment based on the use of a dynamic geometry software.

In the following, three main trends in learning geometrical reasoning will be discussed: reasoning in building proofs, reasoning in geometry from context and visual reasoning. Duval's three functions of reasoning – *proving, expanding knowledge* and *explaining* will serve as a framework for analyzing geometrical reasoning in each of the above.

1. Reasoning in building proofs

Several chapters and sections in this volume discuss in depth the role of deductive proof (e.g. Chapters 5 and 6). It seems that there is a consensus that deductive reasoning (or in the classroom jargon 'proving') still has a central role in geometry learning. However, the classical approach is now enriched by new facets and roles.

Consider the two classical aspects: deductive reasoning as part of the human culture to be learned by human beings, and deductive reasoning as a vehicle for verifying geometrical statements and showing their universality.

For generations, geometry was taught as *the* context for teaching deductive reasoning and was governed by the above classical aspects. The most sound claim against this view is that the product – a written proof – was more important than the process of proving, and thus teaching tended to neglect both the visual geometrical context (shapes and relations between them) and the learner. Nowadays, research and development effort is being invested in order to create innovative learning environments that still regard deductive reasoning as a basic element to be learned. However, these learning environments try to take into account the pupil's point of view by designing learning situations which help pupils *feel* an intrinsic need for explanations, and thus provide the invitation to appreciate the strength of deductive justification as an explanatory tool, or even to attempt to produce them.

The most important curriculum efforts in this spirit are those based on dynamic geometry software. (For a detailed analysis see Hoyles on "proof in dynamic geometry contexts" in chapter 4, and Jones in this chapter).

A main pedagogical feature of these learning environments is that pupils are partners in the discovery of geometrical facts and in the reinvention of geometrical relations, by exploration and inductive reasoning. The aspects suggested by Duval of seeing reasoning as an *extension of knowledge* and as an *explanatory tool*, come to life in classroom reality using these learning environments. By experimentation and inductive generalization, pupils extend their geometrical knowledge about shapes and relations and extend their 'vocabulary' of legitimate ways of reasoning. Deductive reasoning becomes then a vehicle for understanding and explaining *why* the inductively

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discovered conjecture might hold. Moreover, deductive reasoning becomes a means to convince others of the validity of the conjecture discovered (de Villiers, [1]).

2. Reasoning in learning geometry from context

The learning activity in dynamic geometry environments is one trend that demonstrates the 'democratization' of reasoning in geometry learning. A second trend is what is called reasoning in learning geometry from context. According to this view, geometrical knowledge can and should be constructed in a meaningful way in contexts that can serve as "fields of experience" (Bartolini Bussi & Boero, section III) or as "geometrical springboards", (Lehrer & Romberg, section IV). The contexts should be 'realistic' for pupils, where realistic is taken in a broad sense. Gravemeijer ([3]) describes the 'realistic' conception of the Dutch curriculum (which holds for others as well):

> ... realistic refers to what is experientially real to students, including mathematics itself. Once the students have mastered some mathematics, mathematics itself can become a 'realistic' context.

There are now many curricula and research projects for whom realistic contexts have different meanings. Each has its own special features in theory, development and research. In section III, Bartolini Bussi & Boero describe their on-going work in Italy, where they designed "fields of experience" based on "phenomena which are remarkable to the history of culture". Gravemeijer [3] and section 3.3 in Hershkowitz, Parzysz & van Dormolen ([6], pp. 176 - 193) discuss the theory and practice of the enormous amount of work in geometry done within the frame of the Dutch realistic mathematics education program. In section IV of this chapter, Lehrer & Romberg illustrate by example the development and research work of "geometry from context" in the USA.

There are some common crucial features to all of these approaches. A key characteristic is what Gravemeijer [3] calls "reinvention through progressive mathematization". The pupils are confronted with situations in which they observe and solve problems in a realistic geometrical context, and investigate the invariants of geometrical figures and relations under realistic changes. In this interaction with the context they mathematize, namely they construct higher mental actions. *Mathematizing* is seen as a human activity, as a sort of organizing process by which elements of a context are transformed into geometrical objects and relations.

Internalization in which the learner is going through "the transformation of external activity into internal activity" (Wertsch & Stone, [11], p. 162) is an important aspect of mathematization. Bartolini Bussi & Boero speak about "the evolution of the pupil's internal context, through the activity

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developed within the field of experience". They claim that in this transition, geometrical knowledge, constructed as a 'tool' in a specific field of experience, becomes an explicit geometrical object which can be implied while interacting with another field of experience.

Mathematization in geometry requires geometrical reasoning. The different kinds of reasoning and explanations, emerging from the need to act geometrically (to mathematize) in "different fields of experience", are part of the similarities and differences among these geometrical environments. That is the change from "what I see" to "how I see" in accordance with the change of one's (the observer's) position, described by Gravemeijer [3], who invites the learner to make use of geometrical tools reinvented by the pupils (i.e. vision lines and angles).

Bartolini Bussi & Boero describe how pupils explore the shadow phenomenon by making conjectures based on their experience, and how this inductive reasoning offers the 'arguments' for the subsequent construction of proof. The quilt design task, described by Lehrer & Romberg (section IV), exemplifies a related sense of mathematization, in that children's informal knowledge of quilts and their aesthetics about what makes a quilt "interesting" are progressively transformed and re-expressed as the mathematics of the plane. The need for pupils in the second grade to explain to their teacher and their peers what they are doing and why, pushes them to invent a system of notation. This system of notation enables them, in a later stage, to discover and explain many geometrical facts about compositions of transformations on the core squares of their quilt design.

In summary, Duval's three functions of reasoning are well expressed in learning geometry from context. As part of the mathematization, pupils reason and explain while constructing and expanding their geometrical knowledge. Reasoning as proving starts from many sorts of inductive justifications; for example, in the Lehrer & Romberg study, an extended failure to find a counter-example was taken by pupils to be a verification. In general it seems that these justifications push pupils towards more formal proving; e.g., Bartolini Bussi & Boero claim that: "the reasoning that works out the conjectures offers the 'arguments' for the subsequent construction of a proof".

3. On Visual Reasoning

On the one hand, it has been claimed that visualization in mathematics education is now in its renaissance (Zimmerman & Cunningham, [12]). But, on the other hand little pedagogical effort seems to be invested to implement it (see Berthelot & Salin in section V). It could be that the community is making the naïve assumption that human beings are born with visual thinking abilities which are applied when needed, and therefore nothing needs to be done to nurture or develop them. (For more on visual education and its status, see Herskowitz, Parzysz & van Dormolen, [6].)

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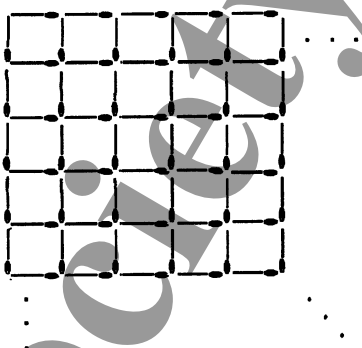
In the following, we revisit some widespread views on visual reasoning, and challenge some of their underlying, or even explicit, assumptions.

– Duval (section II) distinguishes between visual processes and reasoning process, and seems to suggest that those are different categories of thought. He claims that a main function of visual processes is that of *subjective verification.* Even if we agree with this view of visualization (and we raise some questions about it below), many mathematics educators include subjective verification as an integral part of reasoning in general.

– As claimed in Dreyfus ([2]), it seems that visual reasoning has low status. It is mostly regarded as an intuitive, supportive, global and preliminary stage in the reasoning processes in general, which sometimes supports further reasoning, and sometimes hinders it. Using Duval's functions of reasoning, we regard visual reasoning as much more than that: it includes many, if not most of the aspects attributed to other kinds of reasoning, including analytical aspects, even proving. Moreover, visual reasoning can function by itself in order to complete a rigorous mathematical argument or blended with other kinds of reasoning, not necessarily as a preliminary to them (see Jones , section VI).

The following problem (*Matches*, Hershkowitz and Arcavi, in preparation), and the solutions we collected and analyzed, reveal how people use visual reasoning to its full power (Fig. 1).

The following is a square arrangement of matches consisting of 1x1 cells.



• Try to guess the number of matches needed for a square of side 5 matches. 7 matches. Explain your conjectures.

• How many matches are needed for a square of side n matches?

• Explain how you reached your conclusion. Try to find more than one way.

Figure 1

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This problem was the basis for activities with groups of teachers from different countries and cultures. The number and kinds of solutions invented and reinvented, varied from one group to another. Numerical and recursive approaches were applied more than once, but visual arguments were much more popular and successful. The analysis of the following two solutions illustrates some of the visual reasoning used.

1. From one square onwards

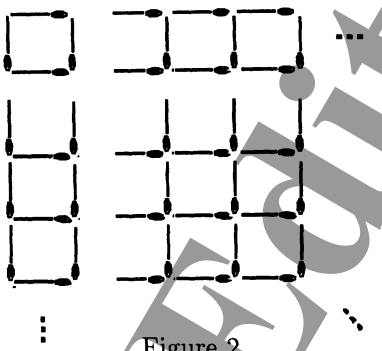


Figure 2

The proposers of this solution saw the whole structure as a combination of different small visual units (see Fig. 2), starting in the upper left corner with a small square (4 matches), and then continuing with units of three in the same row and same column $2 * (n-1) * 3$, and ending with the remaining units of two $(n-1)^2 * 2$. In order to obtain the generalization, these teachers decomposed the whole structure into visual primitives, and then applied an abstract quantitative synthesis in which they combined the visual units and the number of times they appear. This solution approach is analytical in the sense that a whole is constructed from decomposition into small recognizable and countable units, and recomposition (reconstruction) thereof.

Even though the final result is written in symbolic language

$$4 + (n-1) * 3 * 2 + (n-1)^2 * 2,$$

the process of attaining it and the reasoning involved is visual.

2. Cut and paste (see Fig. 3)

Here the solvers 'cut' the big square along its diagonal and then placed the two halves alongside each other (Fig. 3). Each half was seen as a staircase in the rows and in the columns; from 1 match at the top to n matches at the bottom. Going down the stairs they obtained the sum of matches from 1 to n, and the same going from left to right in the columns. Because it is only a half they had to double it. Thus, at the end, they obtained four times the sum of matches from 1 to n. In symbolic notation:

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$$[(1+2+3+...+n)*2]*2.$$

As in the previous solution approach, the visual reasoning consisted of analysis (decomposition into units) and synthesis. However, here, the analysis was preceded by visual manipulation of the whole structure to support the subsequent visual analysis.

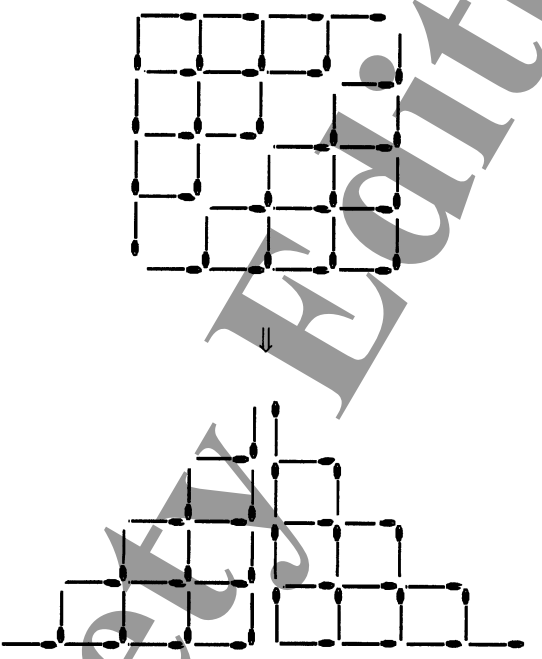


Figure 3

The above examples of visual reasoning are only two of the many solution approaches used. We would like to suggest, that the visual reasoning is much more than the intuitive support of higher level reasoning, it is the backbone of a rigorous proof. The visual process includes: 1) a new way of looking at the situation in order to suggest a generalization , 2) its proof and verification in one process, and 3) an explanation of a 'why' the generalization holds (Hanna, [4]).

Examples like this show that the innate visual capabilities of human beings can be nurtured and put at the service of learning mathematics, not only in a supportive role, but sometimes as the main method of attack.

Acknowledgments: I would like to thank Maxim Bruckheimer, Abraham Arcavi and Joop van Dormolen for their through-reading and challenging comments.

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Section II. GEOMETRY FROM A COGNITIVE POINT OF VIEW

Raymond Duval

Geometry may be exciting for mathematicians and for anyone who likes mathematics. But what about the other people who must learn mathematics in their curriculum? This question arises when we look at the numerous and deep difficulties which the teachers encounter. Teaching geometry is more complex and often less successful than teaching numerical operations or elementary algebra. Therefore why teach geometry to all pupils? This question begs another one: how should geometry be taught? In order to

put forward some ideas on this basic issue we must take into account the underlying cognitive complexity of geometrical activity.

Geometry involves three kinds of cognitive processes which fulfil specific epistemological functions:

– **visualisation** processes with regard to *space representation* for the illustration of a statement, for the heuristic exploration of a complex situation, for a synoptic glance over it, or for a subjective verification;

– **construction** processes by tools: construction of configurations can work like a *model* in that the actions on the representative and the observed results are related to the mathematical objects which are represented;

– **reasoning** in relationship to *discursive processes* for extension of knowledge, for proof, for explanation.

These different processes can be performed separately. So visualisation does not depend on construction: there is access to figures, whatever way they are constructed. And even if construction leads to visualisation, construction processes depend only on connections between mathematical properties and the technical constraints of the used tools. Ultimately, if visualisation is an intuitive aid that is sometimes necessary for finding a proof, reasoning depends exclusively on the corpus of propositions (definitions, axioms, theorems) which is available. And in some cases visualisation can be misleading or impossible.

However these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry.

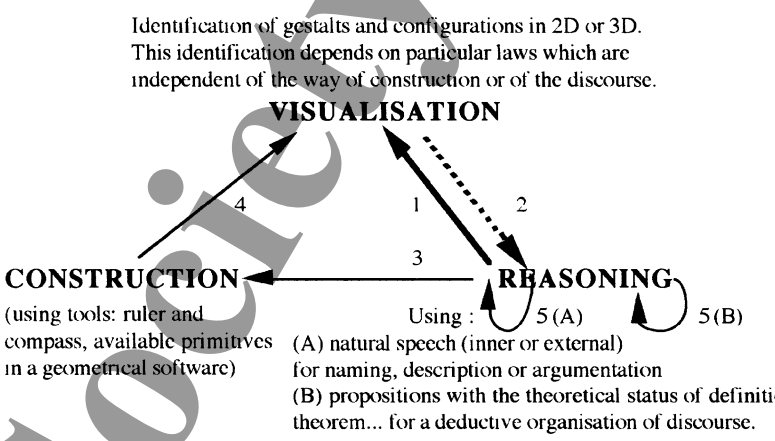


Figure 1

The underlying cognitive interactions involved in geometrical activity.

In Fig. 1 each arrow represents the way a kind of cognitive process can support another kind in any task. Arrow 2 is dotted because visualisation does not always help reasoning. Arrow 5(B) emphasizes that reasoning B

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can develop in an independent way. In many cases we can have a longer circuit. For example, 2-5(B)-3 can represent the way of finding a construction order for a given figure; 4-2-5(A) or 5(B) can represent ways of describing a construction order.

Thus we can see the basic problem of the teaching of geometry at lower and upper secondary schools: how to get the pupils to see the communication between these three kinds of processes? The difficulties raised by proof are well-known, and it seems more natural to favour first the construction and visualisation processes. But that raises the following general question: does practice in one kind of process bring out development for the other two kinds?

Our research has enabled us to put forward the following framework of analysis:

1. The three kinds of processes must be developed separately.

2. Work on differentiating between different visualisation processes and between different reasoning processes is needed in the curriculum, for there are various ways of seeing a figure; in the same way there are various kinds of reasoning.

3. The coordination of these three kinds of processes can really occur only after this work of differentiation.

1. Basic topics in geometry from a cognitive point a view: Vision and Reasoning

Vision: is looking at images and figures enough to see what they represent?

What a figure lets you see is one or several 1D/2D or 2D/2D gestalts (straight or curved line, closed outline of a triangle, of a quadrilateral, etc.) or 3D/2D gestalts (cube, bowl, etc.). Visual identification of these gestalts depends on perceptive organization laws, and all these gestalts can be used to represent real objects or mathematical objects. But in order to represent a mathematical object, a figure must fulfil two specific requirements :

– to be a configuration, that is to be a joint or a merger of several constituent gestalts having relations between them which characterize the configuration (visual condition)

– to be anchored in a statement which fixes some properties represented by the gestalt (hypotheses). This discursive anchor gives the mathematical entrance in the configuration (proof condition). Then a first distinction between two apprehensions of a figure becomes obvious:

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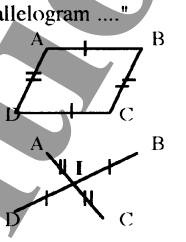
PERCEPTUAL APPREHENSION	DISCURSIVE APPREHENSION of a figure : association of gestalts and statement which determines the represented object.	
I. Visual	(***anchorage change***)	
	II a.Visual → Discursive	IIb.Discursive→Visual
 Identification of a **gestalt 2D**/2D. It can be seen as a roof, as the above part of a table, as a square in another plane than the frontal one, as a parallelogram, etc. Gestalts are more easily seen as geometrical representations *when they are being constructed with tools* (ruler and compass primitives of some geometrical software)	 "ABCD is a parallelogram" In the context of a geometrical proposition, this **gestalt 2D**/2D becomes a *predominant* **configuration 2D**/2D of *several constituent gestalts* **1D**/2D (here lines as the sides of a....). The geometrical representation is given through the relations between constituent gestalts. It is the reason why 2D gestalts are more easily seen as configurations in relation to its construction.The geometrical discursive apprehension involves *a **dimensional change** in the perceptual apprehension* of the **2D**/2D gestalt	"Let ABCD be a parallelogram ...."  Several possible configurations for the mathematical object "parallelogram": relations between segments (the properties of the represented object) are emphasized by dashes

Figure 2

Different entrances in a figure

One sees the significant difference between I on the one hand, and IIa-IIb on the other hand, which are generally confused. In I what is seen is only one gestalt which can show any object: roof, rectangle from a particular perspective... In II the same gestalt must be seen as a configuration of several constituent gestalts, each one representing for itself a segment or a point because the perceived gestalt is stated as a parallelogram. Therefore visualisation in II is quite different from I. In II visualisation requires an internal movement between the predominant configuration 2D gestalt and the constituent 1D/2D gestalts merging into a whole. This internal movement implies a **dimensional change** in the perceptive organisation of the way of seeing. This internal dimensional change must not be confused with the **anchorage change** implied by the two passages IIa and IIb, which are not equivalent.

Internal dimensional change and anchorage change are the characteristics of the mathematical way to look at a gestalt or a configuration.

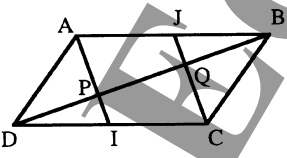
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How does vision work in problem solving: the operative apprehension

In a geometrical figure there are more constituent gestalts and more possible subconfigurations than the ones which were explicitly mobilized for its construction or which are explicitly named in the hypotheses (Duval [5] p.182). It is this surplus that creates the heuristic power of figures: some subconfigurations (or some constituent gestalts) give the key ideas for a solution or for an explanation. From a cognitive point of view this raises the question of the visibility of these relevant subconfigurations: how can they be distinguished?

In order to understand this cognitive process, we shall compare three elementary problems.

Problem 1. *Take a parallelogram ABCD. I and J are the midpoints of CD and AB. Prove that the segments DP, PQ and QB are of the same length.*



Many subconfigurations can be seen in this initial figure. But the following ones must be distinguished and selected for the solution.

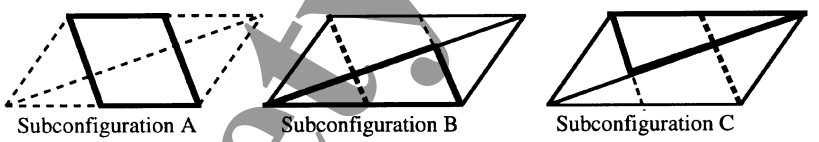


Figure 3

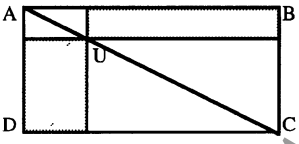
Relevant subconfigurations for the problem 1

Focusing on the relevant subconfigurations B and C requires that one has explicitly thought of a theorem, the theorem of middles. We have observed 13-14 years old pupils which could not see them in the starting figure (even after discussion in order to explain why DP = PQ and PQ = QB). Here, we are in a discursive apprehension with anchorage on propositions (discursive visual): the distinction of subconfigurations is brought about by applicable definitions and theorems. The figure plays only the role of an intuitive aid for the application of propositions.

The situation is quite different in the problem 2 where no explicit knowlege (definition, theorem...) is needed to see the figure and to find the relevant subconfigurations: we can entirely explore the situation either in a visual apprehension, or in a visual discursive apprehension:

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Problem 2. *In the following figure, AC is the diagonal of the rectangle ABCD. Compare the areas of the two grey rectangles, when the point U is moving on the diagonal.*



Many subconfigurations and constituent 1D/2D (or 0D/2D) gestalts can be seen in this initial figure. But the following ones must be distinguished for the solution.



Figure 4
Relevant subconfigurations for problem 2

Subconfigurations A and A' are quickly identified and remain invariant when the point U is moving on the diagonals. Subconfigurations B and B' are respectively included in A and A' and perceptively overlap, whatever the position of the point U on the diagonal. No reference to explicit geometrical knowledge is needed to see this in the initial figure. Here the vision can be the only leading process in the problem-solving. The main difficulty concerns the identification of the subconfigurations B and B' (Mesquita [8]). We have observed that there are several factors which trigger or inhibit the distinction or the visibility of a subconfiguration in a starting figure. The complementarity (yes or no) of the constituent gestalts for the subconfiguration and the convexity (yes or no) of the subconfiguration are among these factors (Padilla [10]). Here subconfigurations B and B' are composed of two non-complementary gestalts and are not convex. Also they seem to be masked by other subconfigurations which are visually predominant. Hence the great difficulty for 12-13 years old pupils and even for older pupils to find them. However, in these two problems no real **figural change** is required, nothing must be added or transformed in the starting figure: all relevant subconfigurations are already given with the starting figure. This is not the case in the following situation, a very well-known proof of the theorem of Pythagoras.

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Prove that in a right-angled triangle $a^2 + b^2 = c^2$.

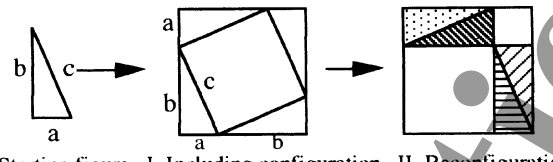


Figure 5
An old and classical proof

The right-angled triangle must be first included in a larger configuration, an external square of side $(a + b)$ with an internal square of side equal to the third side of the right-angled triangle. That is a real figural change. Afterwards this larger configuration can be reconfigurated by changing some constituent gestalts, so that the internal square appears divided into two smaller squares of area a^2 and b^2. The first figural change implies, as in the problem 1, a discursive anchorage. This anchorage is made on the properties stated in the relation $a^2 + b^2 = c^2$. The second figural change depends only on possibilities of reorganisation of the constituent gestalts. Here the constituent gestalts of the including configuration are changed like pieces of a jigsaw, in order to get another configuration which is relevant for the solution. In the context of a given problem one or several particular reorganisations are relevant while the other reorganisations are not. We have called this figural change the **operative apprehension** (Duval [1], [4]). The most interesting and important point here is that operative apprehension can be more or less visible and that its visibility depends on the same triggering or inhibiting factors as for the previous distinction of subconfigurations. It is this figural change, or operative apprehension of a figure, which gives to vision its heuristic power in problem-solving.

These three examples, like any example, are evidently very simple and limited to some particular field of geometrical knowledge. And we can easily imagine the increasing complexity of visualisation with figures representing richer geometric situations. But what we have stressed through these examples concerns the general phenomena about visualisation in geometry. Thus visualisation in geometry implies necessarily at least one of the three changes about what is seen: dimensional change, figural change, anchorage change.

The dimensional change is the most obvious. At least in space geometry where one needs first to distinguish the different possible plane sections of a solid, for example, in order to select the relevant ones. In fact, the identification of a plane in a 3D/2D representation raises a very important

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problem which also concerns the first steps in the geometrical representation of space (Rommevaux [13]). And the change from the sensori-motor perception of a 3D object to its 3D/2D representation is never evident nor immediate: there is a long way which goes through plane representations. Here, we have only explained the dimensional change implied in plane geometry: a 2D gestalt must be seen as a configuration of a 1D or 0D gestalt! In plane geometry, like in space geometry, dimensional change is a basic cognitive process in the way of looking at a figural representation. And we can put forward the thesis that visualisation in geometry fulfils a heuristic role when the mathematical objects or properties which are relevant for proof can be seen in a configuration of higher dimension than the dimension of gestalts or subconfigurations which represent these objects in the starting figure. And in order to illustrate this point, here is the following problem of plane geometry (Mesquita [9], p.135):

Construct a square inscribed in a triangle.

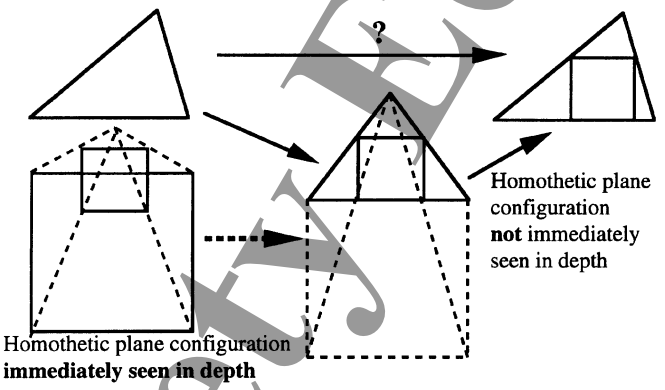


Figure 6

In this problem, finding the detour through the homothetic plane configuration is more natural when it can be looked at in depth (Lémonidis [7]). The dimensional change is like a hidden internal move up and down in the number of dimensions for the visual apprehension of a configuration.

The figural change, or operative apprehension, is the more complex and may be the least conscious. It must be distinguished from perceptual apprehension with which it is connected, and from discursive apprehension (which is anchored on hypotheses and on knowledge of definitions, theorems ...) from which it is completely separated. It concerns specific figural processes. Until now we have identified three great kinds of figural change, with several operations for each of them, and for each operation several factors triggering or inhibiting its visibility (Duval [4] p. 148). That gives the means to analyse the heuristic contribution of a starting figure for a specific problem and to expect the difficulties and the blockages that it can

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generate (Duval [4], 149-154; Padilla, [11]). The figural change is like an action which transforms the visual organisation of a configuration.

The anchorage change in the discursive apprehension is the most familiar. We see and talk (aloud or mentally) about what we are looking at. Thus visual distinction arouses words at least implicitly, and the words mentally uttured can turn the focus of attention towards some unnoticed aspects in the figure. This anchorage change goes often unnoticed. Alas for the teaching of geometry! Because a pupil has not the same *inner speech* about the perceptively identified gestalts and configurations as a mathematician has. And there are relationships between inner speech and reasoning. Looking at a figure can be sufficient to understand a geometric situation or to be convinced *only when all these changes can be performed and are merging.* But are they a natural and common way to look at any figural representation, whatever they are materially or mentally produced? Can I (pupil) see what you (teacher) see without you having to explain me and without you pointing out what I should have seen? That is the question...

What is reasoning in geometry?

The word "reasoning" is used in a very broad range of meanings. Any move, any trial and error, any procedure to solve a difficulty is often considered as a form of reasoning. More specifically any process which enables us to draw new information from given informations is considered as "reasoning". In this way, induction, abduction, inference are various kinds of reasoning.

From a cognitive point of view there are different kinds of processes which depend on the *form* in which information is presented and also on the way information can be *organized* (Duval, [5]).

In geometry the given information, or the available information, is given under visual organisation of nD/2D gestalts and under some semantical networks from which not only gestalts and objects can be named, but also from which questions, hypotheses, conjectures about gestalts, objects and their relations can be generated. And this given information must be processed at a representational and symbolic level, even if some models can be physically constructed. Therefore our question becomes: what are the cognitive processes in geometry involved in problem solving and in proof? We must distinguish three cognitive processes:

(1) **a purely configural process**, described above as operative apprehension;

(2) **a natural discursive process** which is spontaneously perfomed in ordinary speech through description, explanation, argumentation;

(3) **a theoretical discursive process** which is performed through deduction. The experience of logical necessity is closely connected to this theoretical process. This can be performed in a purely symbolical register

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or in the natural language register. But these two registers do not provide either the same difficulty or the same significance for the pupils.

We have already emphasized the fact that operative apprehension is completely independent of any discursive process. This is why visualisation is an irreducible process for research in geometry. But visualisation can be embedded in a natural discursive process (Fig. 1, arrow 5(A)): what is sometimes designated as "figural reasoning" is rather a kind of spontaneous description of a purely configural process. On the contrary, a purely configural process cannot be embedded in a theoretical discourse even if it gives sometimes the key ideas for a proof. And, what is more important, there is a gap between the natural discursive process and the theoretical discursive process (Duval [3], [5]). One of the main problems of the teaching of geometry is the inability to make most pupils to overstep this gap. And sometimes teachers don't get a clear awareness of the significant difference of working between the natural discursive process and the theoretical discursive process. We shall focus now briefly on these two processes.

Reasoning as a natural discursive process embedding a purely configural process

In order to simplify the presentation let us go back to problem 2. A purely configural process enables us to distinguish all possible configurations, even the least directly visible, and to recognize the relevant ones in the context of the problem (Fig. 4). But this is not enough to solve the problem. Operations of another nature are needed. They are shown in the following sequence where the basic discursive operations are marked by bold types.

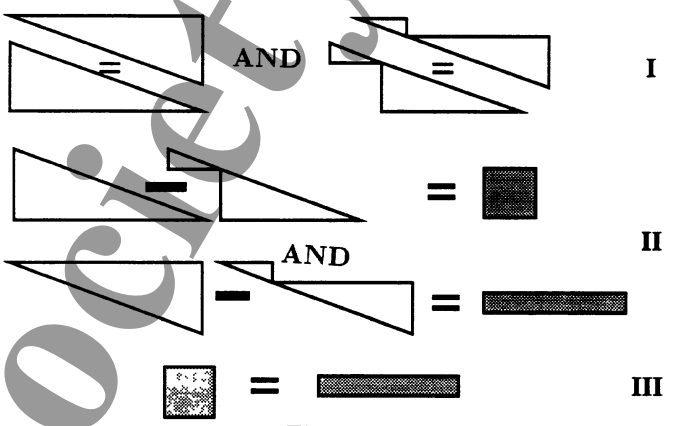


Figure 7

We can observe two levels of organization:

– a global level in three steps I, II, III which look like three statements.

– a local level internal to each step: subconfigurations look like words and the connector "and", as well as verbal abbreviation symbols ("=" meaning

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"produce" and "–" meaning "taken from") are needed in order to organize a step. We must recall that equality between two similar subconfigurations can result from a visual overlapping.

We can make the linguistic presentation of this sequence more explicit, only by describing it. That is what 12-13 years old pupils do more or less precisely through their oral or written argumentation (Mesquita [9]). Evidently other sequences for this problem are possible and can be expressed by other arguments. But we are referred to the same way of reasoning. Here visualisation and spontaneous verbalisation are very close to each other. This is no longer the case with the process of theoretical discourse.

Reasoning as a theoretical discourse with a deductive process

In geometry, to reason in order to prove requires two critical conditions:

(1) to use *propositions*, each one having beforehand a *specific theoretical status*: axiom, definition, theorem, hypothesis, conjecture, etc.

(2) to use *only* theorems, axioms or definitions to take a step toward the conclusion.

In other words, here the form in which information is given is different: it can be only propositions. And the organisation is quite different. We have three levels of organisation:

– a global level in which steps are linked according to their conclusion;

– a local level in which at least three propositions are organised according to their status (hypothesis or previous conclusion, definition or theorem, local conclusion);

– a micro-level internal to propositions used as rules (definitions, theorems ...) in which one must distinguish two parts, the part of conditions to verify and the one of conclusion to state.

There is nothing common between the organisations of the local level in a natural discursive process and in a theoretical discursive process. First, because the propositions are linked according to their status. Next, because this organisation works by substitution of propositions as in a calculation and not by association or opposition as in natural speech (Duval [3]).

That is not a natural way of reasoning. It is very far from the one used in discussion, or in everyday life. To link propositions according to their status goes very often against the spontaneous or evident associations. And axioms, theorems or definitions are not arguments to support a thesis or an opinion: to use a theorem requires first and only as much checking operations as the theorem includes conditions. Many pupils cannot distinguish this theoretical deductive process from the more natural discursive process, even when they mention, apparently in a correct way, definitions or theorems. It seems to them to be a contrived and useless demand from their teachers. But those who discover the deductive process, especially at local and micro-levels of organisation of propositions, make the personal experience of the logical necessity of the conclusion and of the power of this way

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of reasoning. They perceive the nature and the degree of strength of their conviction change (Duval [2], [5], p.224-231).

And now back to our two questions about visualisation and reasoning in geometry

In geometry, visualisation covers together perceptive, discursive and operative apprehension of a figure as a representation of space. And because it does not require mathematical knowledge, visualisation plays a basic heuristic role and, through the operative apprehension, can give something like convincing evidence. What are its relationships with the different kinds of reasoning? In order to give a short presentation, we can start from the two following typical behaviours:

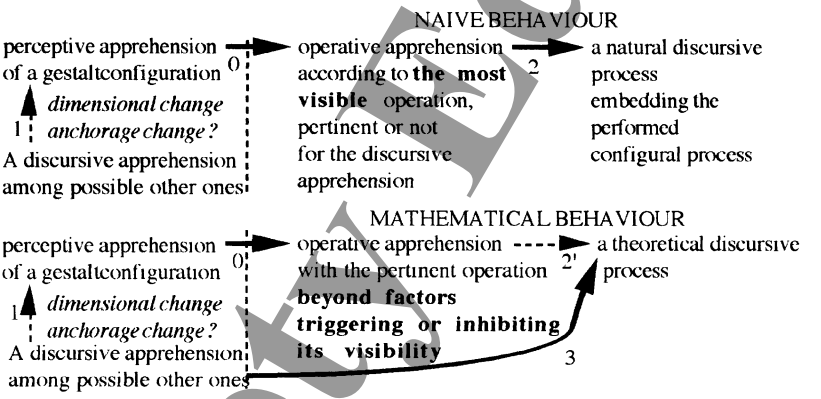


Figure 8
Two typical behaviours.

In Fig. 8, starting from the same geometrical situation (on left), two wholly different behaviours are possible. One reacts at what is spontaneously visible (0) and reasoning works like a description of the steps of the configural change leading to a solution (2). In the other, reasoning starts only from the discursive apprehension and is independent from visualisation (3). The purely configural change does not give the steps and the organization of deductive reasoning for the proof, but it shows some key points, or an idea which allows to select the main theorems to be used (dotted arrow 2′).

Two things must be noticed for our topic:

– For some geometrical situations naïve behaviour is efficient, but under very narrow conditions. The gap between the perceptive apprehension and the discursive one, due to the dimensional change and the anchorage change,

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must be small (arrow 1 on Fig. 8). And factors triggering the visibility of the pertinent operation must be stronger than the inhibiting ones.

– There is a *double gap* between naïve behaviour and mathematical behaviour. The one is about the visualisation and the other is about reasoning.

Thus some specific skills must be developed from the common way of looking at figures and from the natural discursive reasoning. It would be a pedagogical illusion to present mathematical behaviour through the appearance of naïve behaviour (or in continuity with it) because of the visualisation. The main problem for teaching and learning geometry is how to get pupils to step over this double gap.

2. Problems of learning and goals of education in geometry from a cognitive point of view

Preliminaries to the problem

We must *amend three widely assumed principles* in the studies of educational mathematics, through some presentations of constructivism and in some uses of the van Hiele classification.

(1) There is no significant correspondence between the thinking development and the construction of any disciplinary knowledge, and therefore between a level of thinking and a level of knowledge. From a developmental point of view, or from a cognitive point of view, a level of thinking is described in terms of general operative structures, of skills, of limited capacities, and there is the criterion of maturation which corresponds with adolescence. From an epistemological point of view, a level of knowledge is relative to a specific disciplinary field where progress with new notions, new objects, new theoretical organisations is not limited.

(2) There is no developmental hierarchy between the different kinds of cognitive activities: visualisation, natural discursive reasoning, theoretical deductive reasoning, formal axiomatic proof, analytic or synthetic processes. In fact, since the representative level (about the age of 2-3 years) until the most mature levels, we have visualisation, speech, reasoning, analytic and synthetic processes. But the way of working of these different cognitive activities is not the same at each level and becomes more and more complex. And if there are some interactions between them, or even if one seems to predominate over the others in particular contexts, they have their specific and independent development. It is not necessary to repeat here what we have explained about visualisation, which is not only a perceptive apprehension of gestalts. *The development of thinking is multimodal and not unimodal.* This rules out any model of development in which different kinds of cognitive activities would be organized into a strict hierarchy from the concrete to the most abstract, from visualisation to the

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axiomatic rigour. All semiotic representations, even the analogical ones, are already abstract. And very often what one calls 'concrete' is what has become 'familiar'.

(3) Concepts and semiotic representations (figures, diagrams, natural and symbolic languages) cannot be opposed as mental entities versus material entities or as understanding versus communication. Mathematical processes require using different registers of semiotic representations. And from a cognitive point of view developments of thinking and learning are achieved through an interiorization of various semiotic representations. This is why mental representations and material representations cannot be really opposed. They are both semiotic representations. Their difference lies in their mode and in their cost of production, but not in their nature.

The processes of thinking development: differentiation and coordination of semiotic registers of representation

Piaget has stated that the first cognitive development is achieved through differentiation and coordination of schemes [12]. At further levels, the development implies a differentiation of the first semiotic registers, the native language and the iconic representation of shapes, and their coordination. We have emphasized the two gaps between naïve behaviour and mathematical behaviour in geometry. There is no progressive change from one to the other.

For visualisation it is necessary to differentiate between the discursive apprehension and the perceptive apprehension. But the crucial point is the operative apprehension. How to make the pupils learn to see the pertinent configural change beyond factors triggering or inhibiting its visibility? When these factors have previously been identified, this learning becomes really possible and gives rise to various transfers (Lémonidis [7], Padilla [11]). For reasoning it is necessary to make the pupils discover how deductive reasoning gets organised and why it does not work as an argumentation or an explanation in the other fields of knowledge (geology, botany, chemistry, mechanics, history). This organisation is not really visible in the utterances in natural language. But it is in the utterances in natural language that a pupil can become aware of this specific organisation and of its process (Duval [2], [5] p. 217-231). It is the condition for differentiating theoretical deductive reasoning from other kinds of reasoning. And there is nothing formal in this learning. A coordination can occur only when the pupils can make these differentiations.

Goals of education in geometry

We can now come back to our initial question: why teach geometry? There are many good reasons for teaching geometry or for teaching more geometry. There are as many reasons for teaching geometry as there are

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possible applications in technology or in the real world. But these are good reasons only for few pupils, for mathematicians, engineers, etc. And they are not sufficient to make many pupils really learn and understand. There is the cognitive complexity of geometry. Geometry teaching has too often this strange effect of making pupils regress. Many pupils lose the efficiency of the naïve behaviour without succeeding in getting an idea of the mathematical behaviour! In these conditions, why learn geometry when so many other subjects (native and foreign languages, history, sciences) must also be learned? It is the underlying cognitive complexity which provides the basic interest of geometry. The significance of geometry, for anybody who does not plan to become a mathematician or an engineer, is to develop visual representation and reasoning abilities and to favour the synergy of these quite different processes.

And that beyond the particular content of such-and-such knowledge ! Geometry, more than other areas in mathematics, can be used to discover and develop different ways of thinking. That must be an essential goal for the teaching of geometry. But one still has to get a more comprehensive and well-balanced practice of its cognitive underlying processes. That means specific learning situations for differentiation and coordination between several kinds of processes in visualisation and in reasoning are required.

What direction for the future?

We have emphasized the cognitive complexity of geometry. But does the teaching problem change completely with the new computer environment? Resorting to computer constitutes a great innovation for the teaching of geometry. Computers provide enormously increased possibilities of visualisation, particularly through the introduction of the aspect of movement (Laborde [6]). And, because this tool dissociates the moment of intention (the subject has only to choose an instruction in primitives of a geometrical software) and the moment of production which is executed by the computer, it opens like an 'experimental' approach in geometry. Everyone can confront his anticipations with the results on the screen. That is striking for the construction tasks. Hand approximations, even with ruler and compass are no more possible, because some mathematical properties are strongly associated with technical constraints. With computers, a true exploration of geometrical situations is possible, geometrical objects are a little like real objects which can be manipulated. But, at the moment, geometrical software are mainly centered on construction. And if construction gives greater place to the discursive apprehension and to the sequential apprehension of a figure, it does not develop all functions of visualisation, and in particular the 'operative apprehension'.

In order to get to know efficient ways of reaching this essential goal of teaching geometry in primary and secondary schools, much research about

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the deep process of the development and the learning of visualisation and reasoning are still needed. These are the necessary steps to the future.

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Section III. TEACHING AND LEARNING GEOMETRY IN CONTEXTS

Maria G. Bartolini Bussi & Paolo Boero

1. Introduction

The reference to 'real' contexts in the teaching/learning of geometry (and more generally mathematics) has been and still is widespread among mathematics educators in this century. Several reasons are produced to justify this choice, *e.g.*:

- pupils' motivation to learn geometry;
- need to establish links between school learning and everyday learning;
- the conceptualisation of geometry as a 'language to describe and interpret reality' or as a 'structure that organises reality'.

All these are pedagogical, social or philosophical reasons. We claim that each of them can be contrasted by means of different options. On the other hand, as regards research, none of them give tools to investigate:

- potentialities and limits of different 'real' contexts in teaching/learning geometry;
- relationships between geometrical activity within 'real' contexts and within school mathematics.

To deal with these problems we shall propose a theoretical framework which includes cultural and cognitive issues involved in teaching/learning geometry in contexts, and a related classification of contexts and geometrical activities that are developed within them. We shall draw on several case-studies of teaching/learning geometry in contexts, that have been developed by our research teams in Genoa and in Modena, for different school levels (primary and secondary school). The main examples that will be given concern: sun-shadows; representation of the visible world by means of perspective drawing; mathematical machines (i.e. linkages and kinematic geometry).

2. Theoretical Framework

The term "Geometry" will refer first of all to the ordinary geometry content of textbooks of all levels (from compulsory education to university). "Geometry" will also refer to geometrical contents of non-mathematical activities, related (at various degrees of explicitness and awareness) to geometrical conceptualisation.

Regarding the general cognitive and educational issues of this contribution, reference will be made to Vygotskij's seminal work on the relationship between learning and development, the teacher's mediating role and the cognitive functions of semiotic mediation tools.

As far as 'didactic' issues are concerned, we will adapt some concepts developed by the French School of Didactics of Mathematics (e.g. 'tool/object dialectic' by Douady, [15] and 'didactical transposition' by Chevallard, [14]). It is beyond the scope of this contribution to discuss these choices and the consistency of the theoretical framework. Some more details will be given about the notion of 'field of experience', introduced by one of the authors (Boero [5], [6], [8]), with the purpose of conceptually unifying the relationships between geometry and culture within teaching/learning geometry. We shall consider 'fields' familiar to pupils as characterized by their own languages, their own behavioural patterns and their own 'cultural identity'.

The 'field of experience' notion concerns the complex relationship which is developed at school between the pupil's internal context (i.e. experience, ways of thinking and of acting related to a given field of human experience), the teacher's internal context, and the external context (i.e. signs,

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objects, objective constraints specific of a given field). The 'field of experience' notion is supposed to be suitable to deal in a unitary way with either extra-school social contexts (e.g. the social practice of measuring) or physical contexts (e.g. sun-shadows) or contexts from school mathematics (e.g. geometry itself), or even contexts where the above different aspects are strictly interlaced (e.g. cartography).

The issue of the evolution of the pupil's internal contexts will be considered through the activities organized and guided by the teacher within appropriate 'fields of experience'. In some real-word fields of experience, the pupil may acquire geometrical tools and thinking strategies which will be used to think and to act more effectively within the same or within other fields of experience, including (through the necessary teacher's mediation) the geometrical fields of experience.

3. The Analysis of the Relationship between Teaching/Learning Geometry and Fields of Experience

The relationship between teaching/learning geometry and fields of experience can be analyzed from two complementary standpoints.

According to the first standpoint, the attention is focused on the evolution of the pupil's internal context, through the activity developed within a field of experience, under the teacher's guidance. The analysis concerns the progressive enrichment of the set of geometry tools the pupil uses in applied geometrical problem-solving, and the change in conceptualising the situations and the phenomena that are brought into classroom activity.

According to the second standpoint, the attention is focused on the quality of geometrical activity within different fields of experience, concerning culturally and epistemologically relevant aspects of the historical evolution of geometry.

The following two subsections will consider examples of activity in fields of experience according to this twofold analysis.

4. Fields of Experience and the Construction of Geometrical Knowledge

4.1. Everyday Geometrical Experience and School Geometry

Everyday culture includes social practices (e.g. measurement of common physical magnitudes such as lengths, drawing and reading maps, scale drawings, etc.) where important geometrical concepts, properties and strategies are used (frequently as implicit operating tools). There are also tools (referring to the same examples as above: rulers, maps, etc.) whose use – according to social conventions – is rich in geometrical potential.

Referring to these fields of experience, the teacher may introduce out-of-school social practices into the class by using tools and language expressions

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which link school activity with the pupils' out-of-school experiences. By accurately selecting the problem-situations to deal with, the teacher may stimulate pupils to undergo those acculturation processes that usually occur outside school in special environments, and used to occur, at large, in the ages preceding the widespread development of schooling institutions (see Bishop, [4]).

Our studies show that when systematic, pupil-involving and 'realistic' activities are performed at school in a given field of experience, not only is it possible to build up skills for solving simple 'practical' problems, but also to help "theorems in action" (Vergnaud, [22]) to emerge, such as the additivity of length measures (Boero, [9]). It is also possible to favour the development of significant cognitive processes: anticipation, hypothetical reasoning and working out of different types of hypotheses and strategies (Boero, [7]; Ferrari, [16]).

In order that the skills built up within a "field of experience" be recognized and used for a more systematic mastering of the field of experience that one is operating in, and in order to reinvest them in other fields of experience, the teacher must also guide the process of making explicit the geometrical knowledge built-in as a 'tool'. This transition from implicit operative tools to explicit geometrical objects shows some difficulties. In our opinion the main research problem lies in identifying, and correctly managing the differences between school geometrical knowledge and geometrical knowledge related to social practices.

4.2. Geometry and 'Scientific' Conception of Natural Phenomena

We are interested in the role played by geometry in giving scientific interpretations of the natural and social phenomena which make up the world-view gradually drawn up by modern culture and which school should hand on to the new generations. With this in mind, the teacher may propose geometrical modelling of phenomena (*e.g.* sun-shadows) which are remarkable for the history of culture. By contrast with subsection 4.1, school learning in this case cannot be replaced by out-of-school experience, not even at an implicitly operating level. Moreover very often the teacher must oppose the pupil's conceptions worked out within his own environment or personal life story.

Our studies (Boero, [8]; Boero et al., [11]; Garuti & Boero, [18]; [19]; Scali, [21]) highlight:

- the importance of general 'principles' and specific 'phenomenon-conceptions', among the resources pupils draw from, and as part of their way of thinking;
- the importance of the sign systems proposed by the teacher in order to stimulate and allow the transition to a scientific conception of phenomena.

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Research problems concern:

• the nature and origin of the conceptions and principles shown by pupils;

• the teacher's relationship with the pupils' principles and conceptions (what relationships must/can be developed?);

• situations and ways the teacher may choose to mediate geometrical signs, without building up stereotypes replacing phenomena.

4.3. Geometry as a Specialized and Explicit Cultural Experience

We now deal with geometry as part of today's scientific culture, as a specialized activity of mathematicians and as a component of the basic cultural grounding of intellectual classes in modern societies. So geometry may provide traditionally learned people with some reasoning patterns (e.g. 'mathematical rigour') and give some ideas which frame the meditation on intellectual experiences (e.g. 'infinity'). In order for a geometrical field of experience to be developed in the class, the teacher must introduce elements drawn from scientific culture for all the aspects qualifying it in terms of 'cultural specialization' (sometimes causing 'cuts' with respect to everyday culture; see Balacheff, [1]).

Our research studies (Boero & Garuti, [12]; Bartolini Bussi, [3]) and the literature on this subject show that:

• explicit reference to problem formulations or to breakthroughs ascribed to eminent historical personalities facilitates the pupils' distancing from their intellectual work and the general and synthetic formulation of their results;

• the comparison between the texts worked out by pupils and standard geometrical texts (*e.g.* those found in textbooks) pushes many pupils to reformulate their texts or the reference texts in order to have them somehow similar to one another.

Research problems:

• how far can the teacher go in pushing the pupil's constructive activity (personal and social) and when should he start proposing cultural models unfamiliar to the class?

• what are the potentials and the cognitive mechanisms involved in directly using historical sources?

• how can the cultural ground be made ready (and how can suitable situations be created) to allow epistemological obstacles to come out? (cf. Fischbein, [17]).

4.4. Connections between the Various Aspects

First of all, it is to be underlined that, in developmental and functional terms, the different types of relationship, which may exist between geometry

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and culture in compulsory school, can be connected to one another, in some cases even within the same field of experience.

For instance, in the field of experience of "sun-shadows", many pupils can overcome their naïve non-geometrical conceptions of the shadow phenomenon through geometrical schematization; this schematization may then be used to tackle 'concrete' problems within the same field of experience, such as determining heights that are inaccessible by direct measuring; (Garuti & Boero, [18]) and, in a theoretical sense, in building up the field of experience of "rational geometry" (Boero & Garuti, [12]).

Another example: in the 'field of experience' of representation of the visible world, in grades 1 to 3, pupils learn to locate observers in different standpoints, to imagine observer's views and to manage geometrical modellization of reality. This activity opens the way to produce, verify and even prove conjectures on the theorems of representation in grades 4 and 5. We shall come back to this point in the following. Through appropriate didactic planning, the work within real-world fields of experience may supply concepts (usually as 'tools') and skills required to work in the geometrical fields of experience. On the other hand, the activities in these fields of experience are based on forms of reasoning which appear to be rooted in non-mathematical experience and in 'mathematizing' experience. In particular, Boero *et al.* [11] show how some real-world fields of experience offer 'resources' to develop an initial 'hypothesis game' which gradually enables the pupil to culturally master the fields (according to the interpretative models of the different sciences), while allowing him to work out more complex and demanding hypotheses (conjectures, interpretations, etc.) in the same or in other fields of experience (including the geometrical fields of experience).

In the same way, in the fields of experience of representation of the visible world (primary school) and of mathematical machines (secondary school), we have observed that through teacher's mediation pupils can reach an increasingly deeper mastery of the field of experience, by means of the progressive construction of conjectures and verifications (Bartolini Bussi [3], Bartolini Bussi & Pergola [2]); the introduction of tools from school geometry lends different words to 'concrete' reasoning and changes the 'concrete' learning environment into a geometrical context.

5. Geometrical Activities in the Fields of Experience

5.1. The historical construction of consciousness

The forms of activity that can be carried out within fields of experience are manifold. We have already distinguished fields of experience according to the kinds of activities performed:

- everyday practices that embody geometrical knowledge,
- scientific modelization of natural phenomena

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– specialized geometrical experiences.

The shift from rationalization of practical knowledge to specialized geometrical experience requires the intermediate production of a network of cultural artefacts, i.e. geometrical tools to be transformed into objects of geometry. The application of objects from specialized geometrical experience to new fields, from inside to outside mathematics as well, creates a new level of cultural artefacts.

The history of geometry up to modern applications has substantiated the existence of suitable fields of experience, where to develop pupils' activity. The geometrisation of the representation of space (realized by linear perspective for artistic purposes and by technical drawing for technological purposes) has created geometrical objects, such as geometrical transformations in the projective space.

The development of the theory of linkages, as a part of the study of birational transformations, comes from both artistic and technological traditions.

Today's development of applied mathematics shows interesting applications: perspective is used in displays for computer systems (e.g. surgical training programs; facilities design programs for air traffic controllers); linkages are used in robotics.

The quality of the human activity in creating such cultural artefacts can be reproduced in the classroom through the constitution of two fields of experience:

– the representation of the visible world by means of perspective drawing and

– mathematical machines (*i.e.* linkages and kinematic geometry).

Representation of the Visible World by Means of Perspective Drawing

The teaching experiment has been implemented in several classrooms over a period of three years as from grade 3 (and extended also to a few 6th and 7th grade classes).

The field of experience, as it is recognisable by pupils, is characterized by the systematic presence of different kinds of actions: drawing in order to produce illusionistic images; speaking (or writing) about drawing; analysis of images; reading of selected (historical) sources; use of mechanical or optical devices (e.g. perspectograph and cameras); discussion.

The explicit and systematic introduction of discussion is a characteristic feature of the whole project, to create a culture of social construction of knowledge that allows the active appropriation of existing knowledge. In the classroom, the teaching experiment forces the early geometrisation of real-life drawing (which in the past produced the theory of *perspectiva artificialis*). This process draws on the rationalisation of practical knowledge to meet the need of producing an illusionistic image of the visible world and results in working out geometrical tools (Bartolini Bussi, [3]).

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Mathematical Machines

This field of experience refers to the grades 9 – 13 of a science oriented upper secondary school (*Liceo Scientifico*), but experiments have also been carried out for in-service teacher training and during a public exhibition of models, aimed at a wide popularisation of mathematics; moreover teaching experiments have also been planned for adult education, namely university preparation of mathematics teachers.

The field of experience of these teaching experiments shows two specific features (Bartolini Bussi & Pergola, [2]):

(a) the presence of historical sources, to be read under teacher's guidance;

(b) the presence of teaching aids, such as physical large-size models (either statical or dynamical), the so-called *mathematical machines,* that give the name to the whole project.

The pupils use the models available in the mathematics laboratory up to the statement of theorems and development of complex proofs in 3-dimensional geometry. The teaching aids allow consideration of different aspects related to different historical achievements, e.g. the birth of the projective approach in the 17th century (i.e. the origin of geometrical transformations as mathematical objects) and the shift to the study of birational transformations by means of linkages in the 19th century (i.e. the constitution of the study of geometrical transformations in a fully fledged theory).

In these two examples, the historical analysis, that is made explicit to learners too, allows multiple and non-ambiguous interpretations of the same object and helps to locate oneself in the flow of the continuous cultural activity of mankind, by experiencing geometry as one of its constitutive elements.

5.2. The Approach to Proof

We shall now discuss a particular, yet important, problem of teaching/learning geometry: the approach to proof (a more detailed presentation is in Mariotti *et al.* [20]). We shall show the potentialities of contextualized geometrical activity in producing conjectures and justifying them up to the production of geometrical proofs at early school level. Among other things, this will lead us to reconsider the history of geometry as a collection of mechanisms for working out solutions to complex problems.

The first example concerns the discovery and the justification of a property of perspective representation: the centre of a rectangular table is located at the intersection of the diagonals of the image of its top. Under the teacher's guidance, pupils pass from the empirical observation of photo-images to the theoretical justification of the property, by including it into an embryonic theoretical system, where '*straight lines are represented as straight lines*'. This activity recalls the crucial step in the development of

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the theory of perpective, when the empirical rules of real-life drawing were transformed into the geometrical theory of linear perspective (Bartolini Bussi, [3]).

The second example concerns a didactical situation about sun-shadows. Pupils are requested to state (and then prove) a conjecture about the following problem: *'We know that two vertical sticks produce parallel shadows. What about two sticks, a vertical one and a non-vertical one? Are the shadows parallel? Always? Never? When?'* Because of a long experience of observation and modelling of sun-shadows, most pupils succeed in developing various strategies, to imagine parallelism and to detect the condition for parallelism. The reasoning that works out the conjectures offers the 'arguments' for the subsequent construction of a proof. The proof allows control of the countless number of possible cases, by means of a general reasoning (Boero *et al.* [12], [13]). In this case, pupils reconstruct the crucial step of managing 'generic' configurations – an issue discussed by Descartes, Poncelet and later by Lebesgue – by means of a dynamical exploration of the problem, that allows detection of the special case of parallelism.

6. Discussion

According to most people, one main aim of school is to hand over to the new generations the cultural heritage of the past. Today (more often than twenty or thirty years ago) most people consider geometry as a part of this heritage. Yet two questions are still pending:

– what content, methods, geometrical skills are to be selected;

– what kind of didactical transposition is to be carried out to implement the selected geometrical knowledge.

In this contribution we have tried to answer both questions:

• we can select content, methods and geometrical skills functional to the operative and cultural mastery of 'fields of experience', and relevant to social reality, to nature and to mathematics. The choice is not among a great number of items, but is rather determined by criteria of intrinsic and extrinsic cultural importance. For instance, proving theorems in geometry is relevant to approaching a crucial part of the mathematician's work; modelling vision or sun-shadows is relevant to rationalizing the experience of the senses and to reconstruct the crucial steps of the historical development of geometry;

• teaching/learning geometry within fields of experience leads to a recontextualization of geometrical knowledge as a network of tools, according to historical roots and cognitive motivations; yet the recontextualized geometrical knowledge is suitable to constitute a network of objects in the gradual shift to geometrical fields of experience, or to be applied to generate new cultural artefacts.

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In this dialectic between 'tool of knowledge' and 'object of knowledge', pupils become gradually acquainted with some peculiar aspects of geometrical work (and more generally of mathematical work). In particular, the examples of "theorems about perspective" and "theorems about sun" show that conjecturing and proving can be considered by the pupil as an effective way for solving problems from reality on the one hand and as a specialized activity of the working mathematician on the other.

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Section IV. SPRINGBOARDS TO GEOMETRY

Richard Lehrer & Thomas Romberg

Children's everyday activity – looking, walking, manipulating objects, drawing – provides frequent, albeit often unreflective, opportunity to build knowledge about space (Freudenthal [5]; Piaget & Inhelder [8]; Streefland [10], van Hiele [12]). This informal, activity-centered understanding of space constitutes a springboard to geometry if pupils are afforded the means to "mathematise" and so reconstitute spatial concepts rooted first in experience (Gauvain [6]). For example, while walking, children develop ideas about position and direction that can be elaborated mathematically in a variety of ways – as coordinate systems, as compass directions, as maps, or as dynamic Logo models. Our approach to geometry education in the primary grades includes the development or appropriation of curriculum and related materials, like software tools, that help children transform experientially based intuitions into mathematics (and mathematics into new intuitions).

However, to effect change at the classroom level, it is not enough to develop material means, like curriculum and software tools, because teachers must establish a classroom culture that serves to base pupil activity in mathematical thinking, not simply doing (Cobb, Yackel, & Wood [3]; Vygotsky [15]; Wertsch [14]). For example, by the normative practice of highlighting similarities and differences among pupil solutions, teachers help pupils develop strategic versatility in mathematical problem-solving.

These and related classroom practices need to be flexibly adapted to emerging mathematical thinking in the classroom. One of the most productive sources of such flexibility is a teacher's knowledge of pupil thinking (Carpenter & Fennema [1]). Skilled teachers have rich and flexible models of pupil cognition and its typical trajectories of change (Clark & Peterson [2]; Fennema & Franke [4]; Schifter & Fosnot [9]). Teacher models of pupil thinking help teachers recognize "teachable moments" and other worthy landmarks in the ebb and flow of classroom activity.

Teaching and learning, then, are best viewed in tandem. On the one hand, it is important to identify mathematically important ideas and to develop tasks and tools that will make these ideas accessible to children. On the other hand, it is equally important to help teachers understand landmarks in the progression of children's thinking, because without a model of pupil thinking, teachers must rely exclusively on a curriculum that cannot, in principle, be designed to meet the manifold possibilities inherent in any classroom. To establish this linkage between teaching and learning, we collaborated with a small group of primary grade teachers to trace the course of children's development in four related areas of spatial experience in childhood:

- perception of form (leading to the mathematics of dimension, classification, transformation);
- experience of large-scale space, including wayfinding (leading to the mathematics of position and direction);
- drawing (leading to the mathematics of maps and related notational systems for visualizing three dimensions in two);
- counting (leading to the mathematics of measure of length, area, and volume).

Collectively, we conducted "design experiments" that revealed typical patterns and progressions of pupil thinking when pupils were immersed for prolonged periods of time in classroom activity that supported the development of spatial reasoning. Each year, we updated our curriculum development efforts, our models of pupil thinking, and our assessment practices in the light of what we learned. We are currently expressing this knowledge with interactive multimedia technologies to foster the development of community for teachers.

1. Quilt Design in 2nd Grade

To illustrate our basic approach and the interaction between teachers' knowledge of pupils' thinking and children's learning, we report on the development of second-grade children's (8-year-olds) ideas about transformational geometry. Our analysis of children's learning begins with the notion of the teacher as a designer of a learning environment. Coupled with her knowledge of pupil thinking, the elements of design include:

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- the nature of the tasks that teachers ask children to perform;
- the kinds of tools and systems of notation that the teacher either provides or encourages children to invent;
- the kind of classroom norms and standards that the teacher encourages and establishes.

We seek to understand the role that each of these elements play in children's learning.

Here the teacher used the context of designing a *quilt*[1] as a task that would allow children the opportunity to explore important mathematical ideas like symmetry and transformation, and to develop conjectures about how these mathematical ideas might contribute to their aesthetic experience of quilt design (Watt & Shanahan [13]). Quilts are part of American culture and therefore are familiar objects in children's experience. The teacher encouraged children to invent systems of notation for communicating about transformations and their compositions. As we shall illustrate, this effort to invent notations led to some important mathematical discoveries. The classroom teacher prompted children to state conjectures and to support their conjectures with mathematical evidence. Also this classroom norm eventually played a large role in establishing children's expectations about what can and cannot be established by appeal to cases, no matter how plentiful or clearly stated.

We trace the progression of pupil thinking over a five-week period by focusing on 'snapshots' of classroom activity and conversation. We begin by illustrating the nature of the task and then go on to describe some of the landmarks of understanding developed by children during this five-week unit (see also Stron & Lehrer [11]).

Transforming Core Squares into Quilts

Pupils first designed a paper-and-crayon "core square" that was the basic object subjected to transformations to produce a quilt. A core square was composed of an array of squares, each of which was partitioned into two right isosceles triangles. The number of different colors was optional, although initial designs were often constrained to two or three colors. During this phase of activity, the teacher asked pupils to find the number of distinct core squares that are possible if one uses only two colors. For some pupils, the very idea of partitioning the square into triangles was somewhat difficult, partly because the activity presupposes the child's acquisition of diagonality (Olson [7]). For these pupils, other tools were provided, like plastic Polydrons, that circumvent the problem of constructing the diagonal. In this instance, the core square was composed of eight interlocking Polydron triangles.

(1) I.e. a cover for a bed, composed of square pieces of cloth, sewn together.

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After creating the core square, children use isometries of the plane to arrange four identical copies of their core square into '2 × 2 designs' (2 core squares long and 2 core squares wide). The 2 × 2 designs were then composed to create the final pattern for the quilt. Fig. 1 displays a core square and the 2 × 2 design that results from performing different transformations or combinations of transformations on that core square. In the figure, different fill-patterns denote different colors. Each core square is two-sided; the front and back are identical.

Later in the unit, other quilts were designed by using a strip (a row) of core squares and then transforming the strip. Variations in form (more triangles in every core square) and color provided additional levels of complexity as children designed their paper quilts. Computer software was also available that allowed pupils to design and explore the consequences of their design decisions rapidly and easily.

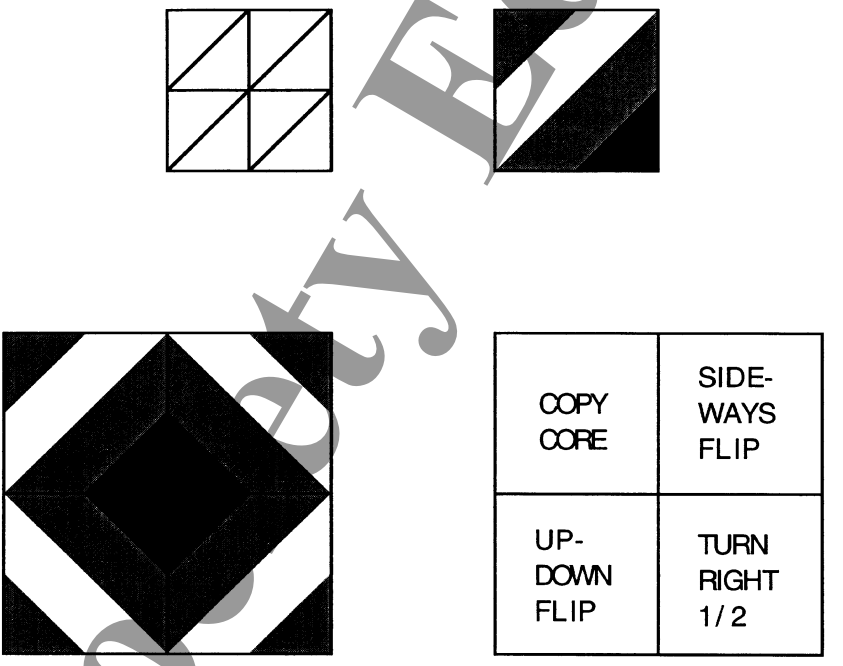


Figure 1

Distinguishing Physical and Mathematical Motions

Because children constructed two-sided paper core squares or core squares composed of Polydrons, they physically enacted flips, slides, and turns. The curriculum confined these initial experiences to translation, vertical and horizontal reflection, and rotation in increments of 90°. To help

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children mathematise these physical motions, children developed a symbol or set of symbols to describe each motion so that they could easily "write directions" for other children to follow when replicating a particular quilt design. Through discussion and consensus building, children developed the following notations for flips: UF for *up-flip* (reflecting about the horizontal axis by grabbing the bottom edge and flipping the square over), DF *down-flip* (reflecting about the horizontal axis by grabbing the top edge and flipping the square over), LF *left-flip* (reflecting about the vertical axis by grabbing the right edge and flipping the square over), and RF *right-flip* (reflecting about the vertical axis by grabbing the left edge and flipping the square over). Note that children's notations for reflections included some that could be distinguished in physical motion but that had no mathematical counterparts. For turns, children decided upon: TR 1 for *whole turn right* (clockwise), TR 1/2 for *half turn right*, and TR 1/4 for *quarter turn right*. Counterparts were invented for left turns as well. Both right and left turns were derived from their previous experience with the Logo programming language and its "turtle" geometry.

In conjecturing about what steps a fellow pupil might have taken to create her 2×2 design from the original core square, some pupils thought that one of the actions taken was an *up-flip*, and other pupils thought that the same action was a *down-flip*. Upon testing these conjectures, the class discovered that both flips produced the same result – it was impossible to distinguish one from another. After the pupils noticed that an *up-flip* was the same as a *down-flip* for that core square, the teacher probed to discover how the class was thinking about the general problem:

T: What do you notice?

S1: They're the same.

T: So an up-flip is the same as a down-flip?

Class: Yes. No. Yeah. No! [No consensus is reached.]

Because no consensus was reached after the first example, the teacher went on to test other examples, using Polydron models of children's core squares.

T: You changed your mind? OK, let's see what we figure out. What we're trying to find out is, is an up-flip the same as a down-flip? OK, this one I'm going to flip up. Watch! [She flips the core up.] This one, I'm gonna flip down.

Class Response: They're the same! Same!

T: Maybe that was just their core square. Maybe they just had a weird core square. Let's try it with this core square. OK, do they look the same, the way I have them now?

At this point, the teacher established a routine for testing the conjecture on a given core square. She placed two copies of the same core square side by side, then flipped one up and the other down. When she finished these

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flips, the core squares were no longer side by side; the one that was flipped up was above the original position; the one that was flipped down, below the original position. It was visually apparent that the actions performed on the two squares were physically different; it was also obvious that both actions resulted in exactly the same pattern on both squares. After several more examples, children became convinced that they needed to change their notation to UDF for *up-down flip* and SF for a *sideways flip*, because different physical actions led to identical results. This process helped children distinguish the plane of action from the plane of mathematics.

The use of notation for communicating design also helped children mathematise composition of transformations. Children again wrote directions for someone else, but a recurrent problem was that children often notated only one of the transformations of a composition. For instance, a child wrote SF for a composed motion, like SF TR 1/4. At first we attributed this to memory demands, but further conversation with children revealed that, for many, the composed action was not visible because their wrists could act in multiple ways simultaneously. After a prolonged effort, the class finally discovered this source of error and revised their notations for compositions. This was another instance of separating the physical from the mathematical and, for some children, of mathematising the action of their wrists. During this sequence, the class also found that order of composition mattered; transformations were not commutative, even though this was a distinction that did not apply to their wrists.

2. Generalization

Case-based reasoning.

The teacher often invited pupils to make conjectures about transformations by considering the generality of a case generated by one or more pupils. The signature phrase, "Do you think this is true all the time?", instigated pupils to consider other cases and to search for counter-examples. The grounds of evidence usually consisted of some larger number of cases that were clearly fit by what the class called a "rule" accompanied by a failure to find counterexamples. This type of conjecture-evidence cycle was illustrated by one child's conjecture about the order of the up-down flip. (The original position was distinguished by a small "x" in the upper left corner, a convention introduced by the teacher to facilitate this discussion.)

T: How many up-down flips could I do, or should I do on Katie's core square so that it will look exactly like it does now?

Pa: [looks at core square, but makes no response]

T: If I do one up-down flip, will it look the same?

Pa: [shakes his head, no]

T: So how many should I do? I've got to get it exactly the way it looks now.

Pa: Two? [quietly]

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T: Two? Let's try it. Watch. Memorize Katie's core square. This is what it looks like. One. [Teacher flips the core square.] Two. [She flips it again].

Br: Yep! Brrrr-di-doo-doo! (trumpet sound)

T: You know how I can tell it's in exactly the same position?

St: 'Cause there – the two green triangles are at the top.

T: Yeah. And, there's her little x, to mark the top of the core, so I know this isn't the flip side. So two up-down flips gets it back right to where it started from.

At this point, children had established that the order of the up-down flip is two. Pupils suggested that 0 and 4 flips would also work, and they tested their conjectures. Br then suggested that any even number would have the same result:

T: Um, what do you think about this idea of Br's? Br's idea is that: I ...

Br: Can't do eleven, but you can do twelve.

T: Meaning two, four, six, eight, ten, twelve. And she kept going. Any even number of flips, and it would look the same.

Br: One's odd; two's even; three's odd; four's even.

T: And then she said for you, 'One's odd; two's even; three's odd; four's even.' OK, that's her idea. Ke has a question for you, Br, about your idea.

Ke: Well, you can go besides by one's, by two's. But if you go by one's it'll just, like, the square will be on the other side; by two's, you could go up, like, as far as you wanted and it would still be the same as when it was started, if you go by two's. If you flip it two times.

T: So if I flip it two times, what will happen, Ke?

Ke: It will be the same.

T: OK, so you're saying I could do what Br was saying, count by two's, as high as I wanted.

Ke: If you stopped at any number, but you counted by two's, then it would be the same as it is now.

T: So, no matter how big that number got, if I just counted by two's and then stopped at that number, and then I flipped it that many times, it would look just like this?

Class: Yeah. Yes!

The class went on to test their conjecture for a series of even numbers and for a series of odd numbers before reaching a consensus that the conjecture was true for any even number.

The limits of cases.

Although the class norm for evidence about a conjecture consisted primarily of generalization from positive and negative cases, the limitations of cases were also discovered by this class. Here, the teacher noted that all children had designed at least one asymmetric core square, and all of these asymmetric core squares were used to create at least one symmetric 2×2 design.

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T: Can you always start with a core square that's not symmetrical, but always find a way to make a 2 × 2 that is symmetrical?

Class: No, no, no.

T: None of you showed me that. Every one of you was able to. Every one of you took every core square you had that was not symmetrical and you made a symmetrical 2 × 2 design. Every one of you.

Despite appeal to pupils' experiences and their cases, many pupils in the class remained unconvinced about the generality of the conjecture. The teacher decided to probe children's thinking about the number of cases that might serve.

T: So if we shared twenty or more ways together today that you could start with an asymmetrical core square, and still every time make a symmetrical 2 × 2 design, how many more do you think we'd have to test before we could say you could always do it?

Ni: Hundreds of hundreds of thousands.

Na: We would have to test all the core squares in the world.

T: Could we do that?

Class: No, no.

Na: We'd have to test all the core squares in the world that are asymmetric.

T: Oh, OK. So we wouldn't have to test all the core squares in the world, because we weren't thinking about symmetrical core squares. We were only thinking about asymmetric core squares. How would I know when I had tested every asymmetric core square in the world?

Children decided that they would have to exhaustively test every case, so the teacher probed further to see if children would find this feasible.

Na: No, because there is a lot of, um, quilts in the world, and there's even more 2 × 2's than quilts, and there's more core squares than quilts in the world. So we would probably have to test. We would probably have to go from state to state and we'd have to go like to every place. We would have to like, um, we would probably have to search, like, from since we were babies!

T: What if I went to every – what if were possible for me to go to every state, every country, all around the world. How would I know when I had them all? Even if I had visited every single place imaginable, how would I be sure that I had all the core squares? Is there only a – is there a certain number of core squares?

Br: No, there could be – people are probably making some right now!

T: Do you think anywhere in the world that we're gonna find an asymmetric core square that can't make a symmetric 2 × 2 design?

Na: We may.

T: Why do you think so?

Na: I don't know. Because some people just make some up, and they'd not be symmetrical, and may have tried all of the 2 × 2 designs that they could make with them and not have them ... be symmetrical.

CC: So you think there are core squares out there that are not symmetrical, ones that can't – no matter how long you think about it or try – be turned into symmetrical 2 × 2 designs?

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Class: Right, yep.

Having found that children find the idea of exhaustive search logically possible but not pragmatically feasible, the teacher tried to appeal to children's classroom experience. However, most of the children were not persuaded by this appeal.

CC: None of those (asymmetric core squares that cannot be transformed into symmetric 2 × 2 design) were here today. We tried over twenty different core squares and they all made symmetrical 2 × 2 designs.

Br: But there may be some in the entire world that can't be.

Na: And there are tons and tons of core squares. Some people may be making core squares right now!

In summary, the class decided that to ascertain the truth of the conjecture by testing examples, they would need to test every asymmetric core square in the world. Then they realized that this would be an impossible task, because they could never be sure that they had tested all of the asymmetric core squares – there might be some somewhere that could not be made into symmetrical 2 × 2 blocks. The pupils struggled with the philosophical concept of induction, and concluded that they could not be sure that the general statement was true by testing the examples. We believe that reasoning about the limits of induction, a theme which resurfaced later in this classroom, sets the stage for proof as a form of argument. We are currently working with teachers on forms of instruction that build on this foundation.

3. Summary

In the United States, instruction about geometry in the elementary school is conspicuous by its virtual absence. We have embarked on a program to redesign geometry education in the primary grades, based in part on outgrowths of children's activity and in part on helping teachers develop understanding of the growth of children's reasoning about space. Viewing teaching and learning as simultaneously developing processes suggests that researchers need to develop theoretical lenses for observing interactions between classroom practices and conceptual change. To this end, we found that everyday activity provided an excellent springboard for mathematising space. However, much of the spring in the board could be attributed to teaching practices that included selection of tasks, tools, and guiding conversation, all of which evolved in light of the teacher's knowledge of pupil thinking.

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Section V. THE ROLE OF PUPILS' SPATIAL KNOWLEDGE IN THE ELEMENTARY TEACHING OF GEOMETRY

R. Berthelot & M. H. Salin

One of the most important challenges of the 21st century is that everybody in the world should be able to get at least a level of knowledge which could be compared with the current compulsory school curriculum of the societies of our western world. Therefore we need to throw some light on the main difficulties met by teachers and pupils during the period of compulsory school attendance (CSA). Our research focussed upon the teaching and learning processes of elementary geometry.

This contribution especially concerns the teaching of spatial knowledge and of plane geometry to French pupils under 13 years old. On the basis

of the research that we have conducted for about ten years, it sets out to present the direction of this research, some results, our methodology and two examples which can help the reader to get an insight into our problematic.

1. On the aims of the teaching and learning processes of geometry

We consider that, during the CSA period, the teaching of geometry must assume the two aims of mathematics education: mathematics as a basic service subject and mathematics for its own sake.

Mathematicians argue that (Euclidean) geometry is the (main) theory of space.

Mathematics curricula for the compulsory period are designed as if the mathematical point of view about geometry teaching could solve by itself all the main spatial needs of useful spatial representations in our society.

For 20 years, many French researchers have studied various didactic phenomena in the teaching and learning processes of geometry. These studies have made explicit the difficulties met by secondary school pupils in the learning process especially. But, regarding the teaching process, we are now convinced that the treatment of spatial abilities is the main source not only of many of the further learning difficulties met by secondary school pupils, but also of some of the main deficiences in spatial representation needs in everyday life.

2. Results and proposals

Our main results can be summarized as follows:

• Because of the constraints due to the size of the different spaces in which daily life interactions take place, natural knowledge of space is strongly structured into three main representations: *microspace* (corresponding to the usual prehension relations), *mesospace* (corresponding to the usual domestic spatial interactions) and *macrospace* (corresponding to unknown city, maritime or rural spaces...). In consequence, the space representation produced by the usual out-of-school experiences is not naturally homogeneous, and is quite different from elementary geometry.

• In traditional teaching of geometry, the usual learning of geometric activities requires of pupils many skills from normal microspace relations; these relations are essential to the teaching process.

• In elementary school, when pupils work with geometrical drawings upon paper, then, to save intellectual effort, they naturally use their microspace representation instead of some geometrical knowledge. Consequently, such tasks may hide from teachers some of the obstacles to the learning of basic geometrical concepts; among these are angles, or space as a set of points, or the recognition of certain configurations that arise in geometric theorems.

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• Concerning pupils over 12 years old, we have good reasons to expect that one of the main sources of learning difficulties in geometry is the previous treatment of geometrical figures on paper during elementary school. These difficulties come from the way that plane figures are considered in the teaching process, as if they were real objects. When teachers and pupils communicate about drawings on paper, they can show apparent understanding, the former thinking in terms of geometric concepts, the latter in terms of objects.

• We have good reasons to expect that geometric knowledge isn't spontaneously transferred to solve space problems. This lack of transference generates deficiences in the control of spatial abilities, even in low-level professional activities.

• We found how some of the obstacles to learning geometry in the primary school can be overcome by a partial blocking of the influence of visual knowledge of microspace.

Our theoretical analysis makes us pessimistic about the viability of local changes, if these are not linked with essential changes in the space constraints of the pupils' tasks. The need is to prevent the pupils from resorting to some aspects of their personal representation of microspace.

In France, if a teaching process were to take place in the geometry curriculum, which aimed at an improvement of pupils' spatial knowledge, there is no chance that it would last unless is aims are not explicitly allocated into the curriculum.

If our results were confirmed by other studies, the consequence for the teaching of elementary geometry would be important and ought to be deepened by larger-scale researches.

3. Theoretical frame and methodology

Some researchers study relations between pupils' difficulties in recognising the useful geometric shapes that are needed for finding geometric demonstrations, and their visual skills in shape-recognition (Hershkowitz [10], Cordier & Cordier [6]). Others (e.g. Duval [7]) direct their investigations toward linguistic and cognitive analysis.

With the "figural concepts" theory, Fischbein [8]) explores the links between the perceptive representation of figures, and the logical constraints of conceptual work with the corresponding mathematical objects.

We chose to explore more precisely the effects on the teaching and learning processes of geometry, of the spatial knowledge acquired through interactions within daily life.

Our theoretical frame is based on a didactic and epistemological point of view introduced by Brousseau [5]): the methodology of the theory of didactic situations. Thanks to this methodology, he and G. Galvez [9]) have specified and extended Acredolo's concepts of "Small and Large-scale

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space" [1] thus leading to the new concepts of micro-space, meso-space and macro-space.

We have analysed some of the specific constraints inherent in geometry, which have a bearing both on the specific didactic contract in the classroom, and on the changes in curricula of the last 50 years.

Didactic engineering is also an important element of our methodology, thanks to the facilities of the Observation Centre of the Bordeaux 1 University (COREM). We have tested our findings by carefully conceived and observed courses. For example (Berthelot & Salin [4]), we chose the concept of sector angles. We studied the difficulties in comparing angles as they appeared in the teaching/learning processes of 12 years old pupils. These difficulties can be interpreted as a didactic obstacle, resulting from both the traditional teaching process, and the influence of the microspace representation involved. A new teaching process was developed, consisting of a few lessons to pupils 10-11 years old, which was very closely controlled by our theoretical frame, and prevented pupils from using their microspace representation. We then showed that the comparison difficulties had disappeared.

4. An example of deficient space knowledge in daily life: the use of plans and maps

4.1 Some results of Grecia Galvez

Too many pupils were getting themselves lost in the streets of Mexico. So, in 1980, the town council launched a program to teach pupils how to find their way.

G. Galvez [9] carried out her research with this problem. She showed that people have (at least) one macro-spatial conception which articulates in a specific way some of the knowledge used when dealing with unfamiliar large-scale spaces. She also showed that this macro-spatial conception is heterogeneous with the two other types of spatial conception (meso-spatial and micro-spatial).

4.2 Some of our results

We extended this work on spatial conceptions, towards geometric concepts.

The primary school must develop the use of maps. What is the knowledge to be taught? We found that pupils learn with success to read names of places, of rivers and other useful information on maps. But, what about the precise abilities necessary to use your plan efficiently when you are lost in an unknown town, or if you have to go through a town to reach another one?

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Our methodology led us first to identify the types of situations in which it is necessary to use plans and maps; then, by analysing the spatial interactions which are necessary to solve these, we identified the knowledge which is necessary in these situations to use plans in order to find and communicate one's way. Lastly we have worked on the learning of those abilities which allow people to orient a plan in accord with their surrounding space.

• *Comparison between the performance of pupils and adults*

We made a qualitative comparison between the results of 10-11 years old pupils and those of adults with only low-level qualifications, reported by Rachedi and Weil-Fassima [13]). The subjects had to find a room after it had been located for them on a plan by the experimenter. The subjects could move and turn the plan, if they wanted; to succeed, they needed to turn the plan through 180°, physically or mentally. Comparison of the results led us to conclude that for this kind of ability, pupils' (aged 10-11) and adults' skills are probably similar, and don't necessarily improve with age.

• *Why don't most of the pupils improve this kind of knowledge about orientation, at school?*

One can argue that most teachers teach how to use plans, making pupils link real space with its paper representation, whether inside the classroom, inside the school or around it. We think that, in their particular environment, pupils have a good knowledge of each of the two spaces (on paper, and real space). In a familiar space, their mental representations of space make them able to get understandable information easily from maps, and to process this data in order to act in space; the link between the paper and the mental representations of the mesospace is easily and implicitly made. But it is quite different in an unknown situation, of which you have no previous mental representation at all. To find their way, people have to take on board the links between the orientation of their body, that of the plan and that of real space, and sometimes the orientation of the persons they are asking their way from – and this during the whole time they are moving.

• *What are pupils taught in mathematics courses about plans?*

It seems that the teaching of geometry in the compulsory school system doesn't improve orientation abilities. When we studied the geometry curriculum, we found that the only abilities which are still taught nowadays in the last years of primary school, are those implied in arithmetic; for example, most of the time, pupils are trained to calculate distances measured on maps, real distances, and the scale.

In every class, the mathematical teaching of how to use space representations very often limits pupils' interactions with real space to interactions with drawings on paper. We made a survey taking into account the last

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fifty years of mathematics curricula in France. Our study shows that the teaching of spatial knowledge which can't be easily expressed by Euclidean elementary geometry has been progressively excluded from the various curricula.

• *Is it possible to improve pupils' control of map or plan orientation, by teaching it during elementary school?*

In 1991, we set up a teaching experience, limited to four or five sessions with a whole class of 8-9 years old pupils (CE2) in the COREM. One year later, we compared these pupils (CM1, 9-10 years) with pupils on the next level who had not been taught this kind of ability. We found a significant difference between the two samples, and a much better result from the youngest ones. Even then, only a little more than 50% of the experimentally taught sample succeeded in this test.

This experience shows us that this kind of ability can be improved by teaching, but that the task is not easy.

5. The example of the rectangle: a plane geometric figure taught in elementary school

The analysis of the traditional geometry teaching process shows various internal malfunctions, in the treatment of geometric figures, for example. Several of these malfunctions have been precisely located by many researchers (notion of angle, the punctual conception of the 2D figures, 3D representations, and so on).

For example, Parzysz' research into 3D geometry has shown that pupils fail to take into consideration that 2D sketches representing 3D situations must only be taken as partial information. They are a mere representation, and not to be regarded as the object itself; but pupils tried to use these figures, to gather all their information (by visual and measuring processes).

The question we now ask is: With what spatial ability and conceptual knowledge are pupils provided by the teaching/learning processes of geometry on paper?

5.1 Experiences

Since 1986, our researches bore on the pupils' capacity to transfer, on their own, their geometry knowledge from paper sheet to real large-scale space.

We chose to base our test upon the rectangle, using scale space a bit larger than the pupils' height. We first tested a hundred 10-11 years old pupils, in teams of two.

A school bench (about 0,5 m$\times$ 2 m) was situated in one corner of a large classroom. The four legs of the bench ends touched the ground in four places, the positions of which were those of the four vertices of a rectangle;

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but this information was not given. The pupils' task was to foresee the positions of the four bench legs, if the bench were to be moved to the opposite corner, and mark them on the ground with chalk. (A material constraint prevented any parallelism with walls or furniture). The pupils could use any geometrical blackboard tools (ruler, square, chalk...). They could do whatever they wanted except moving the bench to the definitive place before finishing the task. Then, in order to verify, they were invited to carry the bench to the predicted positions. After that, they were asked what they would do if they had another trial elsewhere.

We found that after one trial, only 50% of the pupils were able to explain how to succeed in this task. They said that they used the right angles to control the orientation of the measuring lines they used, so as to foresee the position of the four legs of the bench. The other pupils only thought about lengths (and parallelism). We have been told that the task was too difficult because the bench legs needed 3-dimensional representations...

Last year, we did the same experience with a smaller sample (35) of 9-10 years old pupils, but individually, with a gymnasium mat instead of a bench; the pupils had to anticipate and mark on the ground, with four little sticky tack discs, the places where the four corners of the mat would be, if the mat were moved to another place in the room (in the same conditions of non-parallelism). The results were similarly bad, even worse. Most of the pupils didn't imagine they had to control the right angles, even after a trial and a verification by putting the mat on the foreseen places. They only used lengths.

5.2 Interpretation

Pupils have been taught to draw rectangles on paper sheets, and to speak about their properties. But they never had to actually use the geometrical properties of angles. Why? Because they use other properties to control their drawings. For example, they can control the shape by perceptive means which are very effective in this microspace. To solve our problem, the pupils had to transport themselves out of the usual space on paper, which is the symbolic space of geometry, into a space where they can't control shapes at first sight nor can they modify them immediately. They had to reconstruct the whole geometric model: lines joining two points (the positions of legs or corners), and control the relative position of lines with angles; they also had to control the links between the space and the model. They were not able to do it by themselves.

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Section VI. DEDUCTIVE AND INTUITIVE APPROACHES TO SOLVING GEOMETRICAL PROBLEMS

Keith Jones

Approaches to the teaching and learning of a chosen topic in geometry can be located somewhere between what are sometimes perceived as two extremes. One such extreme is characterised as "intuitive"; the other as "formal" or "axiomatic". There seems to be a number of ways of looking at the relationship between these two positions. Piaget ([6] p. 225), for instance, appears to suggest a hierarchy when he writes:

Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense)

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during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).

This perspective of a hierarchy, with a shift, be it sudden or gradual, from "intuitive" to "formal", is long-standing. In the U.K., as long ago as 1923, the Mathematical Association were recommending three stages in the teaching of geometry [7]; briefly:

Stage A: intuitive, experimental work;

Stage B: 'Locally' deductive work in which formal symbolism and deductive reasoning is introduced, but where intuition and induction still have a place and will be used to bridge logically difficult gaps; and

Stage C: Globally rigorous work.

This model, interestingly enough, has similarities to the van Hiele approach which has received some attention over recent years (see, for example, Fuys *et al.* [4]). In the same vein, the US NCTM curriculum standards for school mathematics state that "the study of geometry in grades 5-8 links the informal explorations begun in grades K-4 to the more formalised processes studied in grades 9-12" (NCTM [8] p 112).

Nevertheless, other viewpoints have been expressed. Fischbein [3], p 244, amongst others, for example, suggests either a plurality or a dialectic when he says that:

> The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.

In this section, I consider why people make the decisions that they do when solving geometrical problems. In doing so, I explore the role of geometrical intuition in geometrical problem-solving and provide an example of the interplay between students' intuitive and formal (deductive) reasoning. The research I describe was designed to investigate the nature of the relationship between the formal and the intuitive components of mathematical activity as students were solving a series of geometrical problems (Jones [5]). The episode I relate involves two pairs of recent mathematics graduates tackling a well-known geometrical problem. The students were using Cabri.

Schoenfeld has written extensively about his work with pupils solving (and not solving) the geometrical problem given below (for example, Schoenfeld [9] and [10]). There are four elements in Schoenfeld's framework for analysing mathematical problem-solving and within each it is possible

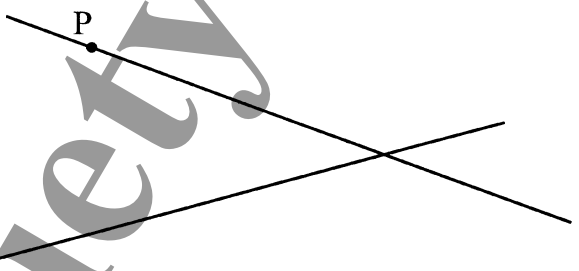
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to suggest a role for intuition. The problem-solvers' resources include intuitive knowledge, but heuristics involve knowing when to use which strategy, and control focuses on major decisions about what to do in a problem, while belief-systems shape cognition, even when the problem-solver is not consciously aware of holding those beliefs. A conclusion Schoenfeld [10] reaches is that, rather than being disjoint, "a deductive approach to mathematical discovery .... and an empirical intuitive approach .. are in fact mutually reinforcing".

It is helpful, at this point, to give Fischbein's definition of *intuition* as a special type of cognition, characterised by self-evidence and immediacy, and with the following properties (Fischbein [2], p. 43-56): intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality, and implicitness. In Fischbein's view, intuitions are theories or coherent systems of beliefs. This conception has similarities to Cooney's [1] idea that the representation of an intuition is likened to a mini-theory, a model, which supports reaching a conclusion, with certainty, on the basis of incomplete information.

Problem

You are given two intersecting straight lines and a point P marked on one of them. Show how to construct, using staightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.



Critical decisions in the solution of this problem are:

1. Constructing a perpendicular line through P

2. Constructing the angle bisector of the angle between the two intersecting lines or constructing a circle centred at the intersection and passing through P to give an intersection with the second line; a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

One subject pair, both male and both with some experience of geometrical constructions, began by reproducing the problem diagram on the computer screen. Their first approach was to construct a circle with a centre chosen somewhere between the two intersecting lines, and with point P as

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the radius point. They then used the facility available with Cabri to drag the centre of the circle so that it appeared also to be tangential to the lower of the two intersecting lines. Though this gave them a solution, they were not happy with this and searched for a way of being 'absolutely sure'.

Subject CR says: "Well, the tangent is perpendicular to the line of radius, isn't it?" so they constructed a perpendicular line through P and constrained the centre of the circle to lie on this perpendicular. Then subject CR suggested that they construct a perpendicular line to the lower of the two intersecting lines and move it into the correct place. At this point, TC wonders if the centre of the circle lies on the bisector of the angle between the two intersecting lines. With that, the problem was solved.

Another subject pair, one female (KH) and one male (KJ), both with some experience of geometrical constructions, used a similar approach. They began by creating the diagram for problem 1 and then proceeded to construct two perpendiculars, one through P and a second perpendicular to the lower of the two intersecting lines. As was suggested but never implemented by the first pair, this second perpendicular line was then dragged into place. At this point, KH says: "I tell you the other thing we could do and that's bisect that angle to find out where they should cross". With that they too had solved the problem.

For both pairs whose methods are described here, once they had solved the problem, they discussed their solution method. This resulted in their drawing up an argument that would properly serve as a proof. In this way, the solution of the problem suggested the structure of a deductive proof. None of the pairs used the alternative method, suggested above, of constructing a circle centred at the intersection and passing through P to give an intersection with the second line. Then a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

For pair 1, the suggestion to draw the angle bisector was made quite tentatively towards the end of their problem-solving attempt:

TC: Yes ... Ah! Now would the centre of the circle lie .. I'm just thinking something slightly different now, because I'm just trying to think, there must be a way of securing the centre accurately .. and I'm thinking .. does the centre of the circle .. sit on the bisector of the angle that's made by those two lines ..

For pair 2 the student was more certain:

KH: I tell you the other thing we could do and that's to bisect that angle to find out where they should cross.

This is how the students accounted for their actions as they watched a video-recording of their problem-solving attempt later the same day. In the case of pair 1:

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TC: .. [long pause] .. well, partly previous knowledge. I wasn't .. completely sure. I wasn't saying 'Oh, yes. This is what does happen'. I just had a sneaky feeling that we were missing something and I couldn't work out what it was, but I thought, well I'm sure the angle .. there must be some connection between the angle between the two lines and the centre [of the circle]. So, let's put the line in, and see what happens.

It turned out to be right, but it was just a sort of stab .. well, it wasn't a stab in the dark completely ...

I can't think why, but I was sure we should be bisecting the angle.

In the case of pair 2:

KH: Ohhh! .. [laughs] .. That's quite interesting because, maybe, .. the fact that there's a cross there [where the two perpendicular lines intersect 'opposite' where the original two lines intersect] actually encouraged me to think: well, we need to know where the cross is going to be. Perhaps if we hadn't have drawn the other perpendicular it would not have come so quickly.

Looking at that picture now I think .. it's ..er .. er .. I mean just having that sort of cross there on the screen opposite the angle there, I mean, that just spells it out. I think perhaps that's why it just came so quickly.

In both cases the students had some difficulty explaining their actions (a well-established methodological issue). Nevertheless, both previous experience and the visual image appeared to play a part in determining the course of action they were suggesting. In this context, Fischbein says: "Experience is a fundamental factor in shaping intuitions" (Fischbein [2], p 85). However, Fischbein (*ibid.*) then goes on to say that "There is little systematic evidence available supporting that view, i.e. evidence demonstrating that new intuitions can be shaped by practice". In terms of the visual image, Fischbein [2], p. 103, claims that visualisation "is the main factor contributing to the production of the effect of immediacy". Fischbein then goes on to relate visualisation to the domain of mental models. The evidence available from this study supports Fischbein's views in the domain of solving geometrical problems.

Further analysis of the data from this study suggests that geometrical intuition has a role in the planning-implementation, and transition episodes of a problem-solving attempt (see Schoenfeld [9], p. 292 for details of these episodes). In addition, it is possible to tentatively identify the following mechanisms as participating in the formation of the subject's geometrical intuitions: premature closure, primacy effect, factors of immediacy (particularly visualisation and anchoring), and factors of globality (see Fischbein [9] p. 204-205 for an explanation of these mechanisms). However, because the analysis examined points of critical decision for the successful solution of the problem, instances of geometrical intuition may, inevitably, tend to form points of transition in the problem-solving process or occur during planning and implementing episodes. The analysis presented here does not consider how intuition may have led the subjects astray.

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The framework Fischbein [2] proposes proved reasonably robust in this study. His problem-solving categories of intuition were identified, and a way suggested to differentiate between anticipatory and conclusive intuitions, in that the subjects' awareness of the critical nature of any decision they make appears to be associated with conclusive intuitions. Secondly, it is possible to tentatively discern the mechanisms that participated in the generation of these geometrical intuitions. The explanations supplied by the subjects in this study provide supporting evidence for these conclusions.

The study described here was designed to provide evidence of particular aspects of the nature and role of geometrical intuition in the process of solving geometrical problems, and of the possible mechanisms that participated in the generation of these geometrical intuitions. The students observed here used a mixture of a deductive approach in, for instance, drawing a perpendicular through point P, and an empirical intuitive approach provided by Cabri: being able to 'drag' a second perpendicular into place(1). Once they had a solution, the ensuing discussion effectively provided a proof. This illustrates how a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems.

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(1) The mediating role of the computer is discussed in Chapter 4.

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3. GEOMETRY IN OUR WORLD

Section. I. GEOMETRY AND REALITY

Joseph Malkevitch

Introduction

There are a variety of ways that mathematics in general and geometry in particular interact with the 'real world'. The purpose of this contribution is to discuss the range of these interactions and to relate them to the teaching of geometry in grades K-12. Obviously, it is difficult to discuss these issues in a way that is equally valid independent of a country's educational system. I will write from an American perspective, but I believe that what I write will be more or less applicable in a variety of countries.

Each of the following interpretations of the phrase "Geometry and Reality" will be discussed in turn:

1. *Geometry of the Physical Universe.* The relation between the physical world in which we live (reality) and the mathematics, more often than not geometry, which is used to describe the physical world.

2. *Geometry and Modelling.* The process by which problems that are posed outside of mathematics (or even within mathematics) are represented in geometrical terms. In this interpretation, Geometry and Reality refers to using geometric models to study real world situations.

3. *Applications of Geometry.* The use of geometry to study specific applications in the real world. These applications of geometry can be organized in terms of specific areas such as operations research, chemistry or image processing or in terms of specific geometric methods (e.g. linear programming or graph theory). The consequences of either point of view for teachers will be discussed.

1. Geometry of the Physical Universe

In the writings (or oral traditions) of the earliest human civilizations, whether in the Western World (e.g. Egyptians, Greeks, etc.) or the Non-Western World (e.g. Mayan civilization, India, Africa or China, etc.), one sees that early in the history of mankind there were attempts to understand

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and organize information about the world in which Man lived. One form this took was in noting the patterns in the motions of the sun, of the moon, and of the stars. Being able to count and to use arithmetic was in part spurred by the attempts to understand the world around us. This attempt to use reason and organized thinking rather than magical thinking appears to have developed independently in many cultures, though 'magical thinking' never completely disappeared and persists even to the present day (e.g. astrology). The tools that early civilizations used to help understand the world in which they lived were very varied. The Egyptians, for example, spurred by a variety of practical problems, pioneered ideas in the development of the computation of the area and volume of various geometric shapes, as well as developing techniques for counting and computation. In the Middle East and West a series of political empires spread and consolidated ideas that were developing in different cultures. Periodically, great consolidations of thought were achieved. One of the most dramatic and historically important consolidations was accomplished by Euclid, whose remarkable book the *Elements* has gone through more different editions than any other book than the Bible.

Unfortunately, little is known about the man Euclid and his goals in writing the *Elements*. Although the *Elements* are organized as a deductive system, starting with definitions and a list of rules (e.g. postulates and axioms), it is unclear whether Euclid was engaged in a purely intellectual exercise or whether he thought that the system he was discussing was a description of the physical universe. With hindsight, we know that no deductive system can, in fact, define all the terms that appear in it. (How much use is a dictionary of a foreign language with no other knowledge of the language it is written in?) However, some modern authors believe that Euclid realized this. According to these observers, his definitions, rather than being naïve, were an attempt to specify the relation between his mathematics and the physical world of points and lines. (In such discussions one quickly has to deal with the issue of whether or not points and lines do or do not physically exist; except as briefly noted below, I will not deal with such interesting but difficult questions.) However, Euclid could have singled out the sphere for more elaborate treatment and there is reason to believe that he studied spherical geometry in detail, as well as that of the plane and 3-dimensional space. After all, he could have made the *Elements* primarily a treatise about the geometry of a sphere. We know that even in the ancient world there were people who seemed to defy the naïve observation that the sun seemed to travel around the earth and that the earth appeared flat! Further, Euclid had a remarkable treatment of the 5 regular or Platonic solids in terms of solids inscribed in a sphere. Euclid chose not to take a spherical geometry approach. However, his remarkable accomplishments have given the *Elements* a time-honored place in K-12. This role has evolved with time and been challenged from time to time.

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Sometimes, however, this role has been very dogmatic. Even as time went on and 'flaws' were found in Euclid's treatment, some people refused to use more modern and mathematically accurate treatments, preferring the original *Elements* instead.

There have been many wonderful milestones in the relationship between the use of geometry (and other parts of mathematics) to understand the physical universe and for geometry itself. These milestones include the saga of specific individuals such as Pythagoras, Thales, Euclid, Archimedes, Galileo, Newton, Bolyai, Lobachevsky, Maxwell, and Einstein. Very recent work, the significance of which it is hard to access, has been done by S. Hawking, R. Penrose, E. Witten, and N. Sieberg. No disrespect is meant by these lists, which do not include innumerable other men and women of many cultures who have contributed to the search for mankind's understanding of the world in which we live. Nor is this saga defined in terms of the accomplishments of specific individuals. It can also be marked in terms of the many intellectual milestones that had to be reached along the way to the remarkable but still incomplete understanding that we have today! Examples from this perspective include the development of a theory of conics, one major contributor having been Apollonius. The study of conics was begun in ancient times, but in the hands of Kepler and Newton resulted in important breakthroughs in our understanding of the world.

Perhaps there is no more dramatic demonstration of the value of mathematics than that a mathematical argument is used to predict something about physical reality which is unsuspected, and this prediction is verified on the basis of the mathematics only. A good example of this in connection with Newton's Laws is that the planet Neptune was located after examination of data about the orbit of Uranus suggested the possibility that another planet was responsible for irregularities in its orbit. In other words, either there was another planet that was affecting Uranus' orbit or Newton's Laws were incorrect. Based on the calculations of the French astronomer U.J.J. Leverrier (1811-1877), the German astronomer J.G. Galle (1812-1910) located Neptune in 1846, the day after he obtained Leverrier's calculations. J.C. Adams (1819-1892), a British mathematical astronomer, had independently predicted Neptune's existence on the basis of mathematical calculation but unfortunately his prediction was not taken seriously by visual astronomers. No one acted on his suggestions that telescopes be used to search for the planet whose presence he predicted. Like Galileo, who also held an academic position as a mathematician though he is much better known for his work in physics, Adams served as Lowndean professor of astronomy and geometry at Cambridge. It is also intriguing that at a later time, Newton's Laws did have to be modified due to experimental work that showed that the orbit of Mercury predicted by these laws was not quite what physically occurred. Only after Einstein's work on General

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Relativity theory was it possible for physicists and astronomers to reconcile the orbit of Mercury with predictions made from Newton's Laws.

Another even more remarkable example of the interaction of mathematics with the nature of the physical world is provided by the work of James Clerk Maxwell. Maxwell developed a collection of equations which govern the propagation of electromagnetic radiation (i.e. light, radio waves, etc.). On the basis of his mathematics he predicted the existence of certain previously undiscovered waves which were similar to light but had a different frequency. These waves were indeed discovered by Heinrich Hertz in 1886. This interaction between mathematics and physics was a remarkable demonstration of the power of a mathematical theory for the description of the physical world. Maxwell's work was important not merely in the realm of intellectual history. The work that his equations set in motion is the basis of a remarkably large part of the technology base for the 20th century (e.g. radio, telephone, televisions, etc.).

Another still later breakthrough for work in mathematical physics was the concept of higher dimensional space. Though we appear to be able to 'see' only three dimensions in space, the realization that mathematically one could have a geometry with any finite number of dimensions and even infinitely many dimensions was a remarkable accomplishment. The value of higher dimensional spaces in exploiting the use of geometry in studying the physical world has developed primarily in the last 100 years. The applicability of higher dimensional space has not been limited to mathematical-physics. For example, in management science and economics one can represent problems using one coordinate for each of many sectors in an economy (e. g. money supply, employment level, etc.). The result is an important role for higher dimensions.

The drama of developments using higher dimensional space is being played out on the eve of the 21st century, with recent developments of geometers, physicists, and mathematical physicists in using string theory to provide a unified approach for understanding elementary-particle pysics and gravity. (Though this topic is too advanced for treatment in pre-college classrooms, teachers and educators can profit from being aware of these developments, and encouraging students to actively read the newspapers for articles describing how ideas from geometry are being used currently to help understand physical space). Intuitively, instead of thinking of space as being made of points, one can think of the basic unit as being a small loop of string. For some time there seemed to be so many string theories that singling one of them out as a description of physical space seemed impossible. But very recent theoretical work suggests ways in which the plethora of riches that is provided by string theory can be organized to get one theory that applies to our physical world. Remarkably, the string theory approach builds in the physics of general relativity. It was this natural way that general relativity had to be built in as a piece of another

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mathematical theory, that led theorists to conclude that they might be on the 'right track' with developing string theory. Examples of this kind in the interaction of mathematics and physics elicited the famous remark of the Nobel Prize winning physicist Eugene Wigner, concerning the "unreasonable effectiveness" of mathematics in physical science.

There is an important question concerning how much of this story – man's attempt to understand his place in the universe by taking advantage of geometric and mathematical tools – should be part of the K-12 curriculum. Part of the difficulty lies in the subtle philosophical issues involved in these discussions, as well as the fact that since geometry is continuing to evolve, if too much attention is given to these matters, little room will be available in the curriculum for more modern issues. As a simple example of the difficulties involved, consider the issue of whether or not the mathematical notion of a continuum can be built into the discussion of lines and curves in geometrical models for space. Do physical straight lines and lines without holes or gaps really physically exist? Such questions make interesting speculation but may not require extended discussion in our classrooms. Yet, students who go on to study calculus will have to come to grips with exactly some of these questions when they discuss the concept of a limit.

2. Geometry and Modelling

There are a variety of ways that geometric tools can be used as an interface with the 'real world'. The branch of mathematics dealing with the process of using mathematics in the real world is called Mathematical Modelling. (To avoid possible confusion, let me note that the term Geometric Modelling (see [2]) has come to have a technical meaning. It refers to the use of various ideas in geometry, often from the theory of splines and algebraic geometry, to assist with the designing of the complex surfaces that must be manufactured, for an airplane, ship, automobile, etc.). In essence, mathematical modelling deals with the process of simplifying real world problems in a controlled manner. By explicitly stating simplifying assumptions about the problem being investigated, one can obtain a clear picture of the extent to which the mathematical model represents the problem at hand. The tools which are available in constructing mathematical models range from matrices, equations, graph theory, differential equations, difference equations, splines, etc. Thus, geometry has a big role in assisting people who construct mathematical models.

Geometrical interpretations are often used within mathematics itself. For example, one way of proving that $1+2+3+...+n = n(n+1)/2$ is by using mathematical induction. However, many consider it an exciting geometrical alternative to use Figures 1 and 2 below, which provide a "proof without words" (see [10]). Note that Fig. 1 involves counting dots while Fig. 2 involves an area argument. Though there is debate about the role of such

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'proofs' there is no doubt that they are convincing for students and often inspire new mathematical insights.

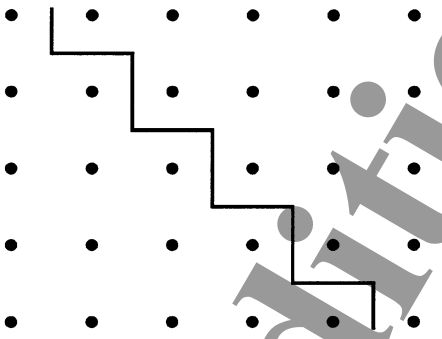


Figure 1
An argument that $1 + ... + n = n(n + 1)/2$.

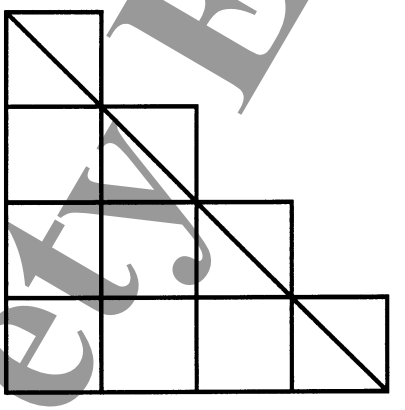


Figure 2
Another argument that $1 + ... + n = n(n + 1)/2$

A surprisingly rich array of such geometrical interpretations of results developed previously only in symbolic form are evolving and belong in the tool kit of ideas that are shown to students in pre-college education.

Similarly, the theory of graphs, a rapidly emerging branch of geometric thinking, provides a wide variety of geometric tools for representing real world problems in mathematical terms. Graph theory is especially amenable to discussions in lower grades. Examples of many applications can be found in [3], and a discussion of the modelling of problems using graphs can be found in [3, 12, 17, 18].

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3. Applications of Geometry

Although historically geometry has proved useful in solving a variety of problems in everyday life and physical science (e.g. finding the amount of carpet to purchase for an irregularly shaped room, or doing calculations in conjunction with putting a new roof on a house), an unexpected recent surprise has been the extent to which geometric thinking has been of value in support of rapidly emerging new technologies. A very brief list of examples follows:

a. *Robotics*

• Ideas from graph theory, Minkowski addition and subtraction (see [22]), and algebraic geometry have proved useful in motion-planning problems for robots. Motion-planning refers to the way that a robot is moved from one location to another in an efficient manner when obstacles are present. (Sometimes a distinction is made between global and local motion-planning. The former refers to moving the robot from one place to another, and the latter refers to the motion of arms of the robot in doing its work when it arrives in the vicinity of where the work it is trying to accomplish must be done.)

• Ideas from a wide variety of branches of geometry have proved valuable in investigating questions concerning grasp and reaching properties of robot fingers and arms. (A typical question about reachability is whether or not a three-linked robot arm can or can not reach a certain point in its work space. A typical question about grasp would be whether or not a three-fingered robot could successfully lift a certain object in its workspace safely or what are the pros and cons of using a robot hand with more or fewer fingers.)

b. *Medical Imaging*

Among the most dramatic changes in the recent practice of medicine is the use of a wide variety of medical imaging techniques. The best known of these techniques is the computerized axial tomography (CAT) scan. A good way of thinking of a CAT scan is that it is a kind of X-ray process that makes it possible to see images of soft tissue rather than merely bone (as happened with traditional X-ray techniques). It turns out that the mathematics necessary to make CAT scans possible was developed by the Czechoslovak mathematician Johannes Radon early in the century. Radon, who was working on a problem in integral geometry, obtained theorems that after the invention of the digital computer made it practical (rather than theoretically possible) to create CAT scan images (see [4]). More recently, PET and MRI imaging also have been developed. Ideas from geometry have also been of assistance in this work.

c. *Telecommunications*

Recent years have seen the development of a large number of rapidly emerging new consumer products from the communications industry. These

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products include fax technology, compact disc technology, high definition television (HDTV), various optical and data storage media, and the Internet. These have been made possible by new mathematical tools, often geometric ones. In a general way, these new developments have grown out of work inspired by Claude Shannon's development of information theory (see [5], [7]). More specifically, these innovations were speeded by the development of codes that are used for data security, data compression (see [19]), and error detection or correction in data. For example, when an image is sent through a fiber optic cable, it involves the transmission of so many pixels that it is important to compress the data before it is transmitted and to uncompress it at the receiving end. Using simple geometrical ideas it is possible to compress images (or text) so that significant acceleration of the transmission of the images becomes possible (see the Data Compression box at the end of this note, pp. 97-98). This offers the possibility of pay-for-view television being transmitted through phone lines and other innovative uses of communications technology. Another application of image compression has been in the storage of finger prints. Rather than storing a photograph of a person's finger prints the image of the finger print is now stored in digital form (i.e. as a sequence of zeros and ones). The form of storage makes it possible to check quickly for a match with an unknown finger print (perhaps of a crime suspect) and to minimize the room needed to store the print.

A good example of the way geometry has contributed to the design of codes is the simple observation of Richard Hamming that if one used binary sequences of the same length d as code words (block codes), the code words can be thought of as vertices on the d-dimensional cube. If one defines the distance between code words as the number of positions where the binary entry in the code words is different (now called Hamming Distance), then the ability of a block code to correct errors can be discussed in terms of this distance. If the spheres centered around the code words are disjoint and have distance $2r + 1$ (or more), then the code will correct r errors, since any received string which has r or fewer errors will be decoded to its closest code word. Although many methods have been developed for constructing error correcting codes, drawing on many parts of mathematics including geometry, the basic theory rests on the simple geometric observation above. Fig. 3 illustrates this discussion in a bit more detail. The Hamming distance between the two code words 000 and 111 is 3. The code words with distance 1 from 000 are 001, 010, and 100, and with 000 make up the "sphere" of radius 1 about 000. The "sphere" of radius 1 about 111 consists of 110, 101, and 011, together with 111, so these "spheres" do not overlap. When only the code words 000 and 111 are used, and no more than one error is made when a code word is sent, any errors that do occur can be corrected, the decoding procedure being: if a binary string is

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received which is not a code word, decode the string to the code word it is closest to, as measured by the Hamming distance.

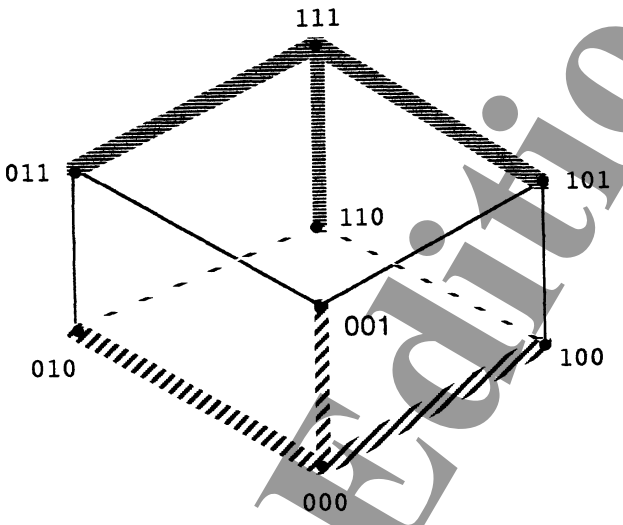


Figure 3
"Spheres" of radius 1 centered at 000 and 111 on a 3-dimensional cube whose vertices represent all the binary sequences of length 3.

d. *Image Manipulation and Processing*

Anyone who has been to a recent movie is familiar with the dramatic ways that image manipulation has progressed from the early days of the cinema. This capability has added to the ways that screen writers and directors can design movies to make them more effective. The family of techniques that has evolved for transforming shapes within images has come to be called morphing. Morphing and other image manipulation techniques have also raised important issues for society. For example, in the popular movie *Forrest Gump*, there are scenes in which the fictional character Gump interacts with historical figures such as President John Kennedy of the United States. Of course, these scenes did not actually occur, but it would be virtually impossible for one to realize this merely from looking at the images involved. Thus, image manipulation techniques have created a variety of legal questions. Photographs are now so easy to alter that it no longer means that in a court of law, what one sees in a photograph really represents a record of what actually happened. Courts have been forced to adjust their rules for the admissibility of photographs into evidence in light of these changes in technology. A more positive consequence of image manipulation technology has been the enhancement of images made, say, by a surveillance camera, to make a crime suspect more easily recognizable. Thus, it is possible to increase the clarity or contrast of an image and in some cases to actually

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remove the blur caused by motion of a human subject against an otherwise unmoving background.

The relationship between mathematics and the issues raised above may not be immediately apparent. However, nearly all the image manipulation described has arisen by the use of computers implementing geometrical algorithms at incredible speeds. In many cases the mathematics involved is very classical mathematics, going back to the 19th century. For example, much of morphing (i.e. the process of changing one shape within an image into another shape) is based on such simple 2-dimensional geometric transformations as translations, reflections, rotations and similarities, and related transformations in three dimensions. By representing such simple transformations in matrix form, it becomes possible to manipulate images or parts of images in what seem to be complicated ways by taking advantage of the speed of modern computers. To these geometrical methods we can add a variety of new geometrical ideas that have been developed for computer graphics applications. In fact, a whole new branch of geometry called computational geometry (see [12]) was born about 1975 in an attempt to develop new geometrical methods and analyze their computational complexity. Among the geometric ideas developed or refined in this emerging area of geometry are methods of finding the convex hull of a geometrical set or computing the Voronoi diagram for a collection of points. (The convex hull of a set is the intersection of all convex sets which contain it. The Voronoi diagram for a set of points consists of determining those points which are closer to one of the points in the set, with respect to a given distance function, than to any of the other points.)

e. *Operations Research*

A variety of geometric methods have found application in operations research. For example, the theory of traversability in graphs (e.g. Euler circuits and Hamiltonian circuits) has found widespread application in such problems as designing garbage collection and postal routes, bus routes, vehicle routing problems, and related questions. In questions of this kind one is trying to find the optimal or best way of providing a certain service. For example, in the school bus problem a city is trying to design routes for its buses that get school children to school as quickly as possible, in a manner that minimizes the cost. (Such models may take into account routes only after the number of buses has already been chosen or the model may deal with the question of the trade-off between the number of buses used and the amount of time necessary to get the children to school.)

Other operations research questions that invoke geometrical ideas deal with coloring problems for graphs. The problem involves finding the minimum number of colors that are necessary to color every vertex of a graph with a different color so that two vertices joined by an edge receive different colors. Imagine one has information about a collection of committees that must be scheduled. If each committee is represented by a vertex of a graph

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and two vertices of the graph are joined by an edge if they have a common member (and hence the committees should meet at different times), then the problem of determining the minimum number of time slots that committee meetings can be scheduled in without conflict reduces to coloring the vertices of the associated graph. More recent applications of problems involving coloring of graphs has been to the assignment of frequencies in the electromagnetic spectrum which can be used for cellular phone transmission in such a manner that calls do not interfere with each other.

In the discussion above we have, loosely speaking, named an area of application and briefly discussed how geometry has proved to be of assistance in the solution of problems arising in this area. However, another time-honored approach has been that of research in an area of geometry for its own sake, with the subsequent development of applications for newly developed theory. An example is provided by the delicate interplay between applied and theoretical developments in network flows and mathematical programming. At various stages this area of mathematics has drawn inspiration from specific applied problems (e.g. vulnerability of telecommunications systems, finding an optimal mix of meats to manufacture the cheapest hot dog, or rerouting of airplanes after a large storm in a certain city has forced their cancellation, delay, or diversion). However, at other times the field has moved ahead, motivated only by internal concerns of the mathematics itself. New and improved theoretical tools at later times were applied in unexpected ways. The mathematics involved in linear programming and network flows has drawn on ideas from many other parts of mathematics: graph theory, linear algebra, analysis, dynamical systems, etc.

4. Geometry and Reality in the Classroom

There is widespread feeling that all of these three aspects (and no doubt other interpretations of the phrase!) deserve attention in mathematics classrooms in grades K-12. What follows are some specific suggestions for how this might be accomplished. Obviously, the examples below are but a small sample of the many ways that the theme of geometry and reality can be raised in the classroom at various levels. Appropriate modifications of the activities listed at a particular grade may be suitable at a higher or lower level.

Grade $K-5$ activities (Ages: $5-10$)

1. Have pupils draw a straight line with a pencil and examine what they have drawn under a microscope or magnifying glass. Also have them examine the straightness of machine-made lines. Goal: having the pupils deepen their understanding of straight and curved, smooth and rough, invariance under scale transformation, etc.

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2. Have pupils use graph-theory models of simple relationships. Example: Draw a graph which represents for a group of pupils the relationship of living no more than 2 blocks from each other. (If a block is not a reasonable standard of distance, one can use some other measure.)

3. Have pupils make a kaleidoscope and study the effects of viewing objects through a kaleidoscope. What would our perception of physical reality be if we had evolved kaleidoscopic vision eyes instead of the perceptual system we have?

4. Have pupils make a cylinder and a Moebius Band out of a long narrow rectangular strip of a paper (say 5 cm $\times$ 25 cm). Pupils should experiment by using scissors to cut down the middle of the cylinder and Moebius Band and discuss the results.

5. Have the pupils take samples of bathroom tile (e.g. the tile consisting of three 1×1 squares, edge to edge in the shape of an L) and explore the kinds of shapes such tiles will fill up.

6. Have pupils measure the angle sum for triangles in the plane and for triangles on a sphere and compare the results.

Activities in Grades $6-8$ *(ages:* $11-13$*)*

1. Have pupils make a variety of surfaces out of paper and investigate questions of straightness and length of path (geodesics) on these surfaces. Good choices are the plane, the sphere, a cube, a cone, and a Moebius Band.

2. Have pupils make all the edge-to-edge tiles that are possible with five 1×1 unit squares attached edge to edge. See if the pupils can show there are exactly 12 such patterns. These patterns are often called pentominoes. Which of the patterns will fold up into a cubical box without a lid? Which of the patterns, with an additional square attached, will fold up into a closed cubical box? Can the 12 pentominoes tile a 6×10, 5×12, or 3×20 rectangle?

Activities in Grades $9-12$ *(ages:* $14-17$*)*

1. Have pupils read a biography of one of the important figures who contributed to the interface between mathematics and our understanding of the physical universe and write an essay reporting on what they learned.

2. Have the pupils study the regular polyhedra using stick and ball models. Which of these polyhedra are rigid and which are not? How many additional rods must be added to the models to get a rigid result? Try to get the pupils to 'discover' Euler's polyhedral formula by doing counts of the number of vertices, edges and faces of different polyhedra.

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Data Compression

Many pupils are ignorant of the methods which lie behind fax machines and compact disk technology. Since any image (sounds are treated in a similar manner) can be thought of as many small cells, called pixels, it is possible to convert an image to a collection of binary numbers. This is done by using a code of binary digits to represent the gray level of each pixel in the image. Rather than use a code which has the same number of binary digits no matter how often or rarely a gray level occurs, a code can be developed where short code words are reserved for gray levels which occur frequently and long code words are used for gray levels that appear rarely. This approach illustrates the idea of data compression.

Figure 4 shows a small picture with 3 different gray levels.

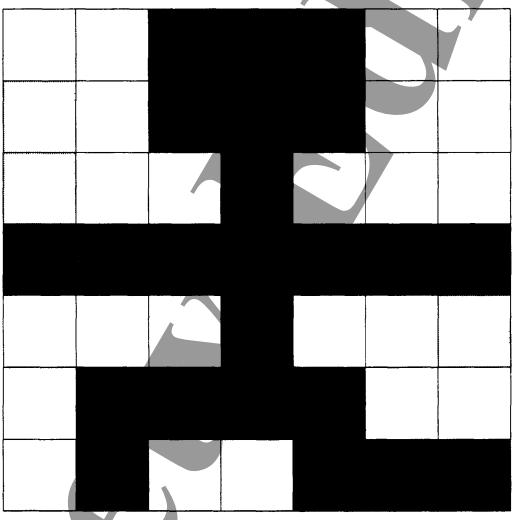


Figure 4

A pixel Martian drawn with three gray levels.

If an image had 6 gray levels, rather than use binary sequences of the same length 3 for the 6 gray levels, we will use a code whose symbols can unambiguously be decoded but for which the average length of the code word needed to send the picture is less than three. This method of developing a code was developed by the American mathematical engineer David Huffman and is now known as a Huffman Code. The simple procedure that Huffman developed is explained with the geometric diagram (graph) shown in Fig. 5.

Sample Solution

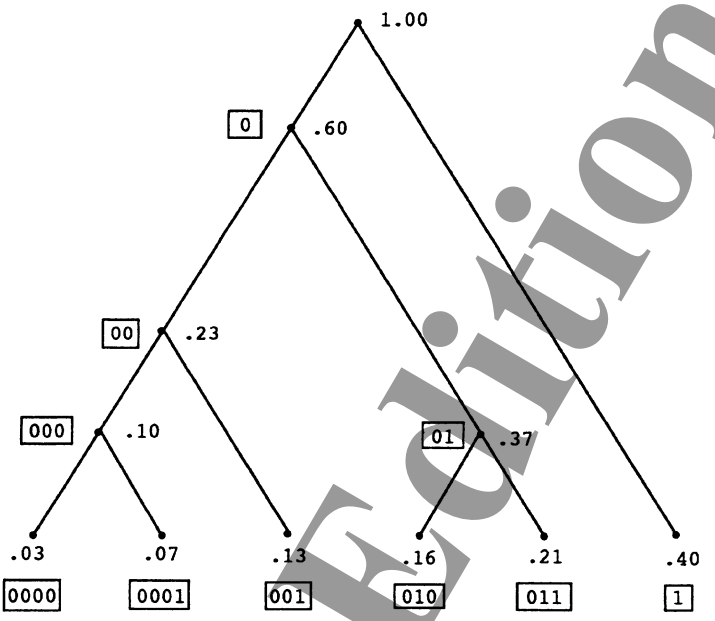


Figure 5

The construction of a binary data compression code that can compress images with six grey levels having the indicated relative frequencies using David Huffman's graph-theory method.

One starts with the relative frequencies of the 6 gray levels listed from smallest to largest. Each of these is represented by a vertex (dot) of a graph. One constructs a new vertex, which is joined to the two vertices which are currently smallest, and joining this vertex to them. A new number is assigned to this vertex by summing the numbers on the vertices used to create it. This process is repeated until a vertex labeled with a 1 is attained. Now a code consisting of a binary string is obtained from the diagram just constructed (called a Huffman Tree). Start at the vertex labeled 1. Each vertex will have two vertices below it and used to create it. The vertex to the right gets a 1 added to the current binary string, and the vertex to the left gets a 0 added to its current binary string. The strings assigned to the original relative frequencies are the code words. Since no string begins with a binary sequence used to encode any of the other gray levels this code can be decoded uniquely.) In this example, if the variable length code constructed is used, then one obtains an average of 2.3 binary digits per gray level (obtained by summing the length of the code words times the relative frequency with which they occur). This contrasts with 3 binary digits per gray level with the naïve approach, thereby achieving considerable data compression.

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Section II. RECENT APPLICATIONS OF GEOMETRY

Walter Meyer

Tremendous amounts of science and technology have been done in the last fifty years. It would be very strange if there were no geometry in all this activity. In fact, there has been lots of geometry, especially in certain areas of computer science such as computer graphics, computer vision, and robotics. For many of these problems, the classical tools of geometry can be used, but there are some problems for which new ideas have been developed. If we want to show geometry as a modern subject, with a future as glorious as in the past, we need to present our students with examples of geometry from the last few decades. Here are some you can have them work on. Solutions are discussed in detail in [6].

1. Scan Conversion

Applying Euclidean geometry on a computer graphics screen is an interesting challenge because here the closest thing we have to points are little patches (Fig. 1) of phosphor-based chemicals which can be turned on in various colors. These patches, called *pixels*, may have irregular shapes, but the eye sees them as tiny points because they are so small. A typical screen might have over 600 rows and 600 columns of pixels. The pixels are evenly spaced so that, if we place a coordinate system on top of them, as in Fig. 1, we find the pixel centers occupying all the points with integer x and y coordinates. What happens if we pick out two pixels and we ask the computer to show a segment connecting the two? What pixels should be lit up to show the segment? How could the centers of these "in-between" pixels be computed from s_1, s_2, e_1, e_2? A variety of algorithms for converting segments, circles and other figures are discussed in Foley, *et al.* [4].

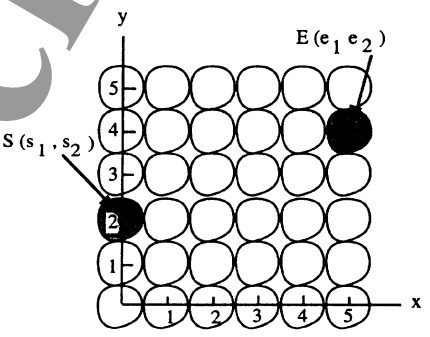


Figure 1

Find a line of pixels connecting the shaded pixels

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2. Rigidity and Architecture

If we take four sticks and hinge them together at the ends, we will be able to change the shape of this quadrilateral by flexing at the hinges.

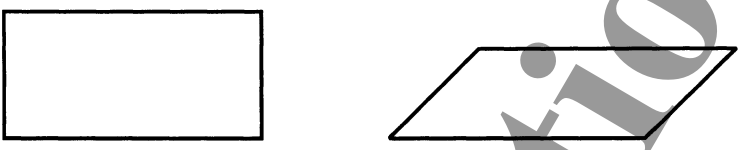


Figure 2
A quadrilateral that flexes at its hinges

Just because a set of line segments has hinges where the lines meet does not mean that the framework can flex. For example, the side-side-side congruence theorem implies that a triangle can't flex even if all vertices are hinged. We say the triangle is *rigid*. A framework of lines with hinges where the lines meet is called rigid if none of the hinge angles can be changed.

The framing of many houses consists of pieces of wood nailed together to form a grid of rectangles. This becomes a kind of skeleton on which the inner and outer walls are attached. Various steps are taken to add to the rigidity, such as extra nails, and the plywood sheets nailed onto the rectangles. Most important for our purposes, a brace can be nailed from one corner to the opposite corner of a rectangle, making two triangles out of a rectangle.

With this as motivation, let us consider the mathematical problem of making a grid of squares rigid by adding diagonals. Perhaps surprisingly, it is not necessary to brace every square. Part a) of Figure 3 shows a set of braces which confers rigidity on the grid (can you see why?). But the number of braces used is not the key factor. Parts b) and c) show bracing that use more braces than in part a), but the braced grids are not rigid.

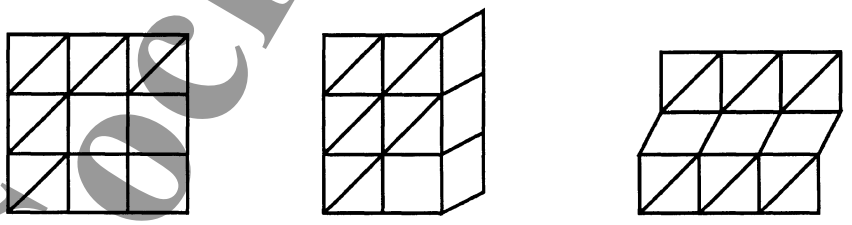


Figure 3
a) a rigid braced grid b) and c) braced grids which have changed shape

Problems

1. Find an algorithm to determine when a braced grid is rigid.

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2. Prove that, in an $m \times n$ grid, if the number of braces is $\leq m + n - 2$, the grid is not rigid.

3. Prove that, in an $m \times n$ grid, it is possible to brace it with $m + n - 1$ braces.

An exposition of this suitable for university and some high-school students can be found in Servatius [9].

3. Fullerene Molecules

In 1985 chemists discovered the buckyball, Chung *et al.* [1]. This is a molecule consisting entirely of carbon atoms arranged in a polyhedron shown in Fig. 4. The map of this polyhedron is exactly that of the traditional soccer ball, with 12 pentagonal faces and 20 hexagonal faces. This was a big surpise because chemists had not thought that carbon atoms could arrange themselves in this way. This discovery stimulated the search for other molecules that followed these rules:

a) The carbon atoms are arranged as the vertices of a convex polyhedron.

b) Each carbon atom is linked to three others by chemical bonds (one of which is a double bond). The three links correspond to edges of the polyhedron.

c) Each face of the polyhedron is either a pentagon or a hexagon. Molecules of this type are called *fullerene molecules* or just *fullerenes*. These names are derived in honour of the engineer Buckminster Fuller, who designed the "Geodesic Dome".



Figure 4

The Buckyball (from Luca Pacioli, De Divina Proportione, 1509)

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All known fullerenes have exactly 12 pentagonal faces. While this may seem to be a quirk of chemistry, it actually results from the underlying mathematics.

Problems

1. Using Euler's formula, prove that every convex polyhedron in which each vertex has 3 edges touching it and each face is either a pentagon or a hexagon must have exactly 12 pentagons.

2. Determine whether there can be a polyhedron as described in question 1, where the number of hexagons is 1. Answer the same question for 2 hexagons. What about 3? 4?

4. Symmetries and Patterns

Fig. 5 shows examples of *strip patterns*. A strip pattern is one which lies between two parallel lines and extends to infinity in both directions. Finite variants of strip patterns are found in tapestries, on walls and circling around pottery, Crowe & Washburn [3]. The patterns of Fig. 5 were found in the pottery of the San Ildefonso Pueblo in the southwestern United States.

Each of these patterns has some kind of symmetry. But what kind? What do you mean by the word "symmetry"? Some would say pattern 7 has more symmetry than pattern 1. How can symmetry be measured? The questions we pose here for patterns are similar to ones which arise, in three dimensional form, in the study of crystallography (Senechal [8]).

A symmetry of a pattern is defined to be an isometry of the plane which carries the pattern onto itself. Pattern 1 of Fig. 5 has translation as its only symmetry while pattern 7 has translation, reflection in both horizontal and vertical lines, glide reflection and a rotation through 180°.

Problems

– Are there any symmetries of strip patterns besides the ones just mentioned?

In studying the San Ildefonso pottery, we find only the seven combinations of symmetries shown in Fig. 5 (adapted from [3], by kind permission of Hadronic Press). For example, no pattern has been found with translation, both kinds of reflections, but neither glide reflection nor a half-turn. If such patterns could exist, then their absence from the San Ildefonso culture might be telling us something about that culture.

– Is there a strip pattern which has the following symmetries: translation, horizontal and vertical reflections, but neither glide reflection nor a half-turn? What about the other combinations missing from Fig. 5?

The solution of these problems leads to the study of compositions of isometries, a concrete and appealing example of group theory.

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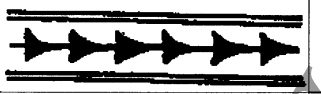
	Pattern	Translation	Reflections: Vertical lines	Reflections: Horizontal line	Glide reflection
1		Yes			
2		Yes			Yes
3		Yes	Yes		
4		Yes			
5		Yes		Yes	Yes
6		Yes	Yes		Yes
7		Yes	Yes	Yes	Yes

Figure 5
Examples of the seven one-coloured strip patterns
These all appear on the pottery of San Ildefonso Pueblo, New Mexico, [3]

5. Robotics and Sensing

A 2-dimensional robot arm has a sensing device at its end which determines that there is an object 30° from the centerline of the second link (dotted) and 5 meters away. In addition, the following information is known to the robot: the lenghts l_1 and l_2 of the links, the angle θ_1 which the first link makes with the positive x axis, the angle θ_2 which the second link makes with the extension of the first link.

Problems

1. What are the coordinates of the object in the coordinate system located at the base?

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2. The robot wishes to change the angles θ_1 and θ_2 so that the object appears 6 meters away and exactly on the center line of the second link. Is this possible? What should the angles be?

These problems can be solved by elementry geometry and trigonometry. But trying to do the same with realistic versions of these problems, involving 3-dimensional robots, quickly becomes tedious and bewildering. More elegant approaches involving isometries and their matrices have been developed (Craig [2]).

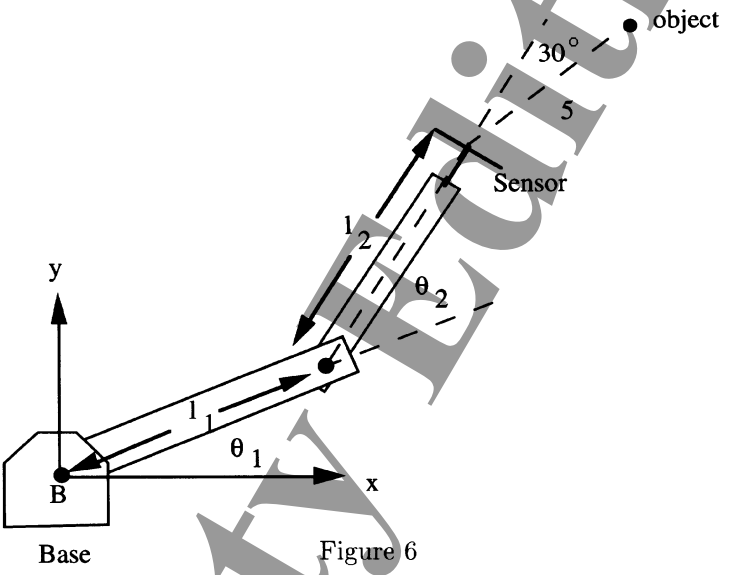


Figure 6
A Planar Robot Arm Senses an Object

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Section III. SCIENCE AND TECHNOLOGY IN GEOMETRY TEXTBOOKS

Peter Legiša

Applications of geometry to scientific and technological problems serve a twofold purpose. On the one hand they show the usefulness of the mathematical abilities acquired. On the other hand they stimulate an interest in science and technology.

It is not particularly easy to obtain examples that are realistic, interesting and accessible to high school students, but here are some that are suitable.

1. Scaling

If we stretch a solid by a factor k, the area is multiplied by k^2 and the volume by k^3. The ratio *area* : *volume* is multiplied by $1/k$. So larger solids have less area per unit of volume than smaller ones. This explains why in cold climates:

i) A small house must have a thicker layer of thermal insulation than a huge apartment building, if the heating bills are to be the same for apartments of equal size in both structures.

ii) Small birds must have a high rate of metabolism, feed often, and/or have to use other tricks to keep warm. For example, wrens (among the smallest European birds) huddle together in caves and holes on winter nights to minimise their outer area and reduce the exposure to cold. Other small warm-blooded animals use similar adaptations.

When concrete was poured into the mould of the huge Shasta dam in Northern California, water cooling through specially constructed pipes was applied since surface cooling was inadequate.

A big heap of coal dust can self-ignite. (The dust particles have an enormous area exposed to the oxygen, while the outer area of the heap is relatively small, so that cooling is insufficient.)

Last but not least, everyone will agree that it is easier to peel a kilo of large potatoes than a kilo of small potatoes.

2. Sound intensity, light intensity, photography

(i) The intensity of light at a point P coming from a point source at O is inversely proportional to the square of the distance OP. The same applies to sound.

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Suppose that the photographer's flash has a range of up to 4 meters. Two flashes of the same kind would have a range of up to 5.6 meters. We would need a flash with four times the energy of the original flash to reach up to 8 meters.

(ii) If the distance from a sound source is multiplied by the square root of 10, then the intensity of the sound is divided by 10, so the loudness fall-off is 10 decibels.

(iii) For simple photographic lenses, the quotient of the focal length and the diameter is the f-number. The aperture mechanism can reduce the opening of the lens and, thus, increase the f-number. The quantity of light coming through the lens is proportional to the square of the diameter of the opening and, thus, inversely proportional to the square of the f-number. The sequence of standard f-numbers or f-stops: 1, 1.4, 2, 2.8, 4, 5.6, 8, 11, 16, 22, 32, 45, ... is a geometric sequence of powers of the square root of 2 (rounded to two decimal places). The amount of light coming through the lens is halved as we pass to the next term in this sequence.

Let us look at a list of photographic lenses with technical specifications. The angles of view can in most cases be calculated from the focal lengths. In fact, for a 35 mm camera consider an isosceles triangle with base c equal to the diagonal of a rectangle measuring 24 mm $\times$ 36 mm. If the altitude h_c is equal to the focal length of the lens, then the angle γ is the angle of view. (Extreme wide-angle lenses are an exception to this rule since they distort the image.)

In the table below we have a list of different eyepieces for a telescope. We can see that:

- the second and the sixth columns are proportional;
- the first and the fourth columns are inversely proportional;
- the fifth column is proportional to the square of the fourth.

We can ask the pupils to find geometric reasons for these relations.

Specifications for each eyepiece lens

Magnification	Real field of view	Apparent field of view	Exit pupil diameter	Brightness	Field of view at 1000 m
15×	3°	45°	4 mm	16	$\simeq$ 52 m
20×	3°	60°	3 mm	9	$\simeq$ 52 m
30×	1.5°	45°	2 mm	4	$\simeq$ 26 m
40×	1.1°	44°	1.5 mm	2.25	$\simeq$ 19 m
60×	0.7°	42°	1 mm	1	$\simeq$ 12 m
30× wideangle	2.4°	72°	2 mm	4	$\simeq$42 m

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4. COMPUTER TECHNOLOGY AND THE TEACHING OF GEOMETRY

INTRODUCTION

Iman Osta

Among all the subjects which are part of the school mathematics curriculum, geometry stands as one which is bound to be influenced most profoundly by the recent progress both in hardware and software development. It is true that to a certain extent all of the mathematics curriculum is affected by the possibilities offered by technological improvement, in particular by the computer capability of handling multiple representations of information (numerical, graphical and symbolic). But the potentialities are most striking in geometry, especially in connection with so-called dynamic geometry software. Explorations can be achieved which otherwise are either inaccessible, or accessible through inordinate efforts.

Our purpose in this chapter is to examine some aspects of the role and influence of computer technology in the teaching and learning of geometry. How can we foresee geometry curricula in the future, taking into account the cultural changes brought by computer technology? What might computer technology offer in regard to teaching geometry? Should educators develop computer packages for teaching/learning geometry to support existing classical curricula with traditional goals and objectives, or should the curricula be adapted to integrate the changes introduced by the usage of computer, and to benefit from the potentials offered by this tool?

While some mathematics educators advocate that the new technology entails new aims and curricula, others don't find it necessary to relate geometry curricula to computer technology. According to the first view, computers are seen as constituents of a new culture involving new modes of work and life, which calls for integrating and infusing their use in schools. As graphic capabilities become more accessible and powerful, students 'do' geometry in a different way, relying heavily on the study of figures and graphics. Geometry is also seen in this context as a modelling tool for astronomical, physical, and other situations, making of it a more practical tool and connecting it to other disciplines.

C. Mammana and V. Villani (eds.),
Perspectives on the Teaching of Geometry for the 21st Century , 109-158.
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According to the second view, computer use should not affect the everlasting aims of classical geometry, or its teaching. It may be used in specific situations as a facilitator or demonstrative tool, it may provide enrichment topics and suggest local additions of geometrical activities which were difficult or impossible before, but all this would not affect the basic goals or methods of its teaching.

While discussing these issues, one should take into consideration the dual nature of geometry:

(1) as the study of space and spatial relationships,
and

(2) as an axiomatic system and a context for deductive reasoning.

These two aspects were once known as respectively the 'practical' and 'speculative' aspects of geometry (quoted by Lindquist, [39] from *Practical Geometry* by Sebastien Le Clerc, 1690). More recently, they were expressed in terms of dichotomies characterizing the origins of school geometry: intuition-deduction, construction-proof, and spatial-numerical (Laborde [34]). How would the use of computer graphics affect the relative status and role of these dual components of the geometrical activity? This question is discussed in the first two sections of the present chapter.

There is a general agreement among educators and teachers that the computer can provide a valuable means for visualizing geometrical situations. Software packages of various types use animation capabilities to provide ways for constructing, moving and rotating configurations, for observing them under various angles, and for modifying some of their features. These demonstrative functions lead to a more functional role of the computer as an explorative tool, making intuition, construction, and spatial sense more important factors, but also providing ways to link them to the theoretical aspects.

Section I of this Chapter deals with visual phenomena in the teaching/learning of geometry, and the links between drawings and the theoretical geometrical objects they represent, using the example of the package Cabri(1). Computers are seen in this section as rich environments for problem solving, and as important components of the *milieu* with which the learner is interacting. Visual evidence is considered to be a catalyst for generating questions and attempts of answers by the solvers, using two types of controls: prior geometrical knowledge and/or empirical experiments with the software. The main question addressed is how to design problem-situations calling for an interaction between visual methods and geometrical methods, which promotes the learning of geometry.

Three types of tasks, with examples, are presented, aiming at promoting

(1) Throughout this book we abbreviate "Cabri-géomètre" to "Cabri".

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the link between visual evidence and geometrical facts. Some characteristics of appropriate tasks are given, in order to foster the use and development of geometrical knowledge, through more functional interaction between visualization and conceptualization. While some of the suggested tasks require the *a priori* use of geometrical knowledge, others seem to motivate the need for a proof.

But how would the status of proof in school geometry be affected by visual evidence? There is a general consensus among mathematics educators that formal proof should remain the backbone of geometry (and mathematics). However, while some of them think that the computer might create obstacles to this aim because of visual evidence and other instruments of validation made available, others think that it can help students see and appreciate the necessity of formal proof, if teachers are skilled enough in developing proper strategies and problems. Others also advocate that at certain school grades, proof can be introduced at informal and less rigorous levels, with the help of computer graphics, in preparation for more formal proofs at higher grades.

In Section II the question is posed, whether the introduction of dynamic geometry systems will promote the transition from informal proof to formal proof in mathematics, or whether it will make it even harder. Will computer use assist pupils in developing a conceptual framework for proof, or will it be seen to replace any need for a proof?

After a brief review of research detecting the problems in the teaching of proof in school curricula, Section II exposes the UK experience with a process-oriented approach to proof in the National Curriculum, where students are encouraged to test and refine hypotheses in order to achieve personal conviction. It is suggested that the computer can offer a new context for designing innovative activities to address the main problem of linkage between empirical experiments and deductive reasoning.

An experiment carried out with 12 year-old students is described, suggesting that the use of a dynamic geometry package such as Cabri, coupled with suitable tasks, may provide an opportunity for some students to develop a basis for a fuller appreciation of the nature and purpose of mathematical proof.

Section III discusses how the computer can help connecting school geometry to professional needs. Indeed, to foresee geometry curricula and aims we have to identify new societal needs, which possibly can be satisfied by a new or modified geometry development through computer procedures. This section advocates the didactical use of CAD software in the teaching/learning of geometry, both as an aim, to prepare qualified users, and as an aid, to overcome some of the problems of visualization and interpretation of 3D figures while teaching geometry.

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Section III also presents an analytical comparison of situations involving graphic representations of 3D objects, using three different techniques: paper-and-pencil, technical drawing, and CAD software. The peculiarities of the CAD situation are highlighted, so as to identify the skills and geometrical content to be emphasized when CAD software is to be used. Two aspects regarding spatial sense are particularly highlighted: the operative processes in the CAD situation versus the figurative ones in a paper-and-pencil situation, and the multitude of possible 3D reference systems inherent in the CAD software, versus the one conventional reference system in classical paper-and-pencil geometry.

Section IV presents newly possible applications illustrating how the computer can serve as a catalyst, providing opportunities for unprecedented pedagogical activities. Three dimensions of geometry teaching are considered as framework for the discussion: mathematics, applications and didactics. Two geometrical examples are given to discuss the crucial role and effect of the computer in reshaping each of the above three dimensions: Splitting and inverting circles (Yin-Yang based tilings of the plane) and Kaleidoscopic effects on computer screens. Along with a third example more related to the dimension of didactics (Symmetric Turtles), these examples provide a support to show types of activities which can be parts of prospective geometry curricula, specifically because this is now made possible by the computer's graphic and dynamic capabilities. Inherent to these activities are relatively complex and difficult-to-realize drawings and geometric figures, which makes their inclusion in school geometry rather difficult, if not impossible, without computers.

We tried in this chapter to include discussions related to several aspects of prospective school geometry in the light of computer technology. Knowing that the subject is wider and more complex than the scope of a chapter, we think that these four sections can provide a global picture of debates and trends, and a projection on possible perspectives for the teaching of geometry in the 21st century. Note that Sections I and II deal with the global goals of geometry curricula, related to the dual nature of geometrical knowledge which permeate its content and teaching, namely visual evidence and proof, while Section III considers its connections to geometrical professional needs and societal goals. A closer and more specific look at the school geometry content is made in Section IV, providing new geometrical activities made possible by the computer capabilities, and studying the effects of such new possibilities on the teaching/learning of school geometry.

For convenience, the references used in the entire Chapter are listed at the end.

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Section I. VISUAL PHENOMENA IN THE TEACHING/LEARNING OF GEOMETRY IN A COMPUTER-BASED ENVIRONMENT

Colette Laborde

Computers can be marvellous tools for providing new visual phenomena calling for mathematical reflection. A necessary condition is, that these phenomena be recognized as such by the learner. This section will deal with visual phenomena in the teaching/learning of geometry for secondary school pupils. It will be illustrated by the example of the software Cabri.

1. Problem situations as learning situations

We assume that problems may play an important role in the construction of mathematical knowledge by the learner in so far as they offer the opportunity to involve his own ideas, and to test their efficiency and validity when trying to find a solution. Brousseau [8] considered the process of elaborating a solution by a learner as the interaction between the learner and a "milieu" which enables the learner to perform some kinds of actions in order to solve the problem, and offers feedback to his actions. In terms of systems, the "milieu" is the system with which the learner is communicating.

Computers are environments in which problem-solving may take place. The graphical and computing possibilities of some software now allow a reification of abstract objects. This applies in particular to mathematical objects. Computers also allow numerous possible operations on these objects, and various types of feedback. But, as Pea [45] claimed, the mediation by the computer provides new tools for operating on these objects and therefore changes the objects themselves. The feedback provided can also be of a different nature. It may be very sophisticated in comparison with a paper-and-pencil situation. All these points imply that designing a problem situation in a computer-based environment requires a new analysis of the mathematical objects, operations and didactic feedback involved in the situation. This might be one of the important reasons for which teachers can be reluctant to use computers in their classrooms; although such an analysis about changes brought by the computer is done in some research, until now it is not really part of the education of teachers.

2. Visual evidence in solving geometry problems

Observing adults with knowledge of geometry, who are trying to solve unusual geometry problems (like the one reported in Laborde & Laborde,

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[38]) in a computer-based environment, shows that visual evidence plays an important role in the solving process:

- visual evidence is interpreted in geometrical terms and generates questions which are solved by means of geometry;
- geometrical analysis triggers new questions which are in a first step empirically explored, giving rise to experiments with the software.

But the fact that the solvers generate numerous questions from the visual evidence is probably due to their mathematical background which enables them to interpret in geometrical terms what they observe.

The question which is addressed here is how to design problem-situations calling for an interaction between visual methods and geometrical methods in order to promote the learning of geometry. Beyond the use of visual phenomena as a catalyst for generating mathematical questions, tasks must be set up which also allow the learner to construct a link between visual and theoretical aspects of geometry.

3. Visual phenomena and the construction of geometrical knowledge in a Cabri environment

At a first glance Cabri is a graphics editor enabling users to draw geometrical diagrams on the screen of a computer. But Cabri is more than a simple editor in that the user can grasp with the mouse an element of the diagram and drag it on the screen: the diagram is continuously re-drawn keeping intact the geometrical relationships which have been declared for its construction, as well as all the geometrical properties implied by them.

Cabri provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle, in a physical sense, the theoretical objects which appear as diagrams on the screen. The behaviour of Cabri is based on geometrical knowledge in two ways:

- diagrams can be drawn, based on geometrical primitives which take into account relevant geometrical objects and relationships. For example, if the menu item "perpendicular bisector" is selected and two points (or one line segment) are (is) designated by means of the mouse, the perpendicular bisector of these points (or this line segment) is drawn on the screen;
- it offers feedback which can distinguish diagrams drawn in an empirical way from diagrams resulting from the use of geometrical primitives. If a parallelogram is drawn on the screen as composed of four segments only apparently parallel, this appearance is destroyed as soon as one of the vertices is dragged (see Fig.1 displaying on the left the original state and on the right the state after point A has been dragged).

The process of drawing a diagram which represents a geometrical object requires communicating to the computer a sequence of geometrical objects and constructions through the selection of menu-items corresponding to the geometrical primitives. From a learning point of view, it is assumed

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that the interaction between these two features of Cabri promotes the use of geometrical properties for drawing diagrams, and therefore the use of geometrical knowledge.

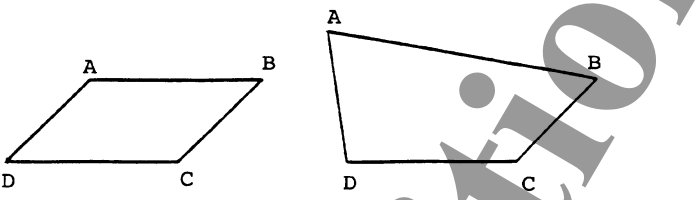


Figure 1

Usually, teaching either confuses *drawings* and the *theoretical geometrical objects* that the formers represent, or creates a gap between drawings and geometry as a theory. For pupils beginning to learn geometry, it may be interesting to present geometrical objects as different from drawings, but also as related to drawings: *geometry can appear as a tool for explaining, producing or predicting the behaviour of drawings.* Teaching may try to reach the following goal: pupils learn to link visual phenomena and geometrical facts, to visually recognize geometrical properties, to interpret drawings in geometrical terms or to construct them. Such learning enables pupils to use drawings or visual representations of geometrical objects as a help in their reasoning at the theoretical level and, through their knowledge, to control the information they take from the visual representations. In this type of activity, they learn to recognize the *necessary* links between visual properties as opposed to those which are only *contingent.* This may be the basis for the construction of proofs, as Hoyles & Jones claim in Section II of this Chapter.

4. Types of tasks promoting the link between visual evidence and geometrical facts

Several kinds of tasks can be distinguished:

(i) moving from a verbal description of a geometrical figure to a drawing; in a paper-and-pencil environment this would be the classical construction tasks in which pupils have to produce the drawings of geometrical objects given by a verbal description;

(ii) explaining the behaviour of drawings by means of geometry, which corresponds to moving from drawings to verbal description and explanation:

- interpreting drawings in geometrical terms; this occurs in tasks in which pupils have to prove why a special property giving rise to a visual evidence is satisfied by a drawing;
- predicting a visual phenomenon such as in problems involving a locus of points;

(iii) reproducing a drawing or transforming a drawing by using geometry.

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We assume that these three kinds of tasks become different in *Cabri* because of the specificities of a Cabri-drawing. Let us examine a priori to what extent each kind of task receives specific meaning in a *Cabri* environment.

Tasks (i)

Producing a Cabri-drawing which preserves spatial properties when it is dragged, requires the use of geometrical properties for its construction, and disqualifies trial and error processes which are only visually controlled. A process of construction of a Cabri-drawing by eye obviously ceases to satisfy the conditions as soon as a basic object is dragged. The use of geometrical relationships is required by the task itself and not by the teacher, as in a paper-and-pencil environment.

It is well known that altering the available instruments in a paper-and-pencil environment may change the conceptual tools (i.e. the geometrical properties) to be used to perform the construction of the drawing. In a software environment it is possible not only to modify the primitives available to the pupils but also to create new primitives which in turn create complex objects. An example of this kind of task follows.

The use of transformations as tools for obtaining geometrical properties

The possibility of configuring the menus and of designing his own menus allows the teacher to ask the pupils to construct geometrical objects by means of specific tools decided by himself. For example, grade 8 pupils have been asked to construct a line parallel to a given straight line (D) and passing through a given point P by using only items among *angle, bisector, reflection, point symmetry* (Laborde & Capponi, [36]). Two solutions are given below (Figures 2 & 3).

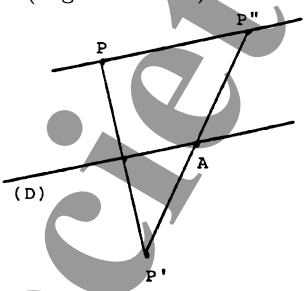


Figure 2
Composition of reflection and point symmetry

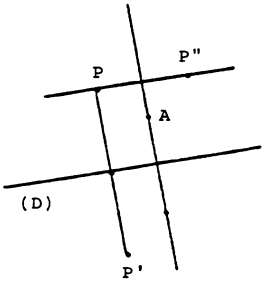


Figure 3
Use of a common perpendicular line

What is new here is the use of a sequence of transformations for visually obtaining a property. Even the fact that reflection may be a way to obtain a line perpendicular to a direction is not always known by pupils because it is not used as such in a paper-and-pencil environment. Guillerault [23] reports that pupils working with *Cabri* were surprised to observe that the line joining a point and its image through a reflection is perpendicular to

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the symmetry line: *"It gives us the perpendicular line ! It must be by chance"*. The use of a composition of point symmetries is usually the result of a theorem; it can be used in a proof but not as a tool for producing a drawing of a figure. Using composition of transformations in a proof is at a high conceptual level and in this case might seem too complicated for the pupils of this age, in comparison with the simplicity of the middle line-theorem in a triangle.

It seems that the computer provides here the opportunity:

• not only to work in the domain of theoretical objects as it does with a proof activity, but also to relate the theoretical concepts to the visual effects;

• to link the visual and the theoretical aspects of this notion of composition of transformations, not only in a passive way (as in a film), but in an operative way (the theoretical objects are used as solution tools of a problem to be solved).

Construction of Cabri-drawings with imposed trajectories

Because the drawing to be produced in *Cabri* is of a dynamic nature, the task can also demand some conditions on its behaviour to be satisfied. For example, construct an equilateral triangle which rotates around its center whenever any of its basic elements is dragged. Fulfilling this kind of condition requires the analysis of the Cabri-drawing in terms of degrees of freedom, which makes such a task different from a traditional construction task. The conditions on fixed and moving points may also require a construction in a different order and thus call for the use of different geometrical properties. This is the case in the equilateral triangle task, in which one has to start verey unusually from the center of the triangle.

Tasks (ii)

Dynamic drawings offer stronger visual phenomena than does one static drawing. A spatial property may emerge as an invariant in the movement whereas this might not be noticeable in one static drawing. In this latter case, it might not be observed, for example, that a straight line always passes through a given point. We assume that in tasks (ii) the software environment gives more importance to the visual observation and therefore may compel the pupils to explain why they obtain such and such visually remarkable phenomena. The software environment may facilitate the appropriation of the explanation or proof task by the child, or in Brousseau's [8] terms it may facilitate the *devolution process* of the problem; the pupil acquires ownership of the task whereas in a traditional environment a proving task can be viewed as a school task with no relation to visual phenomena.

Such tasks have been used with *Cabri*. Grade 9 pupils, for example, were asked to produce the locus of points from which it is possible to view a segment AB subtending a right angle and to try to explain why it is a

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circle (it is possible to verify experimentally with *Cabri* that it is a circle in various ways, using facilities like measuring segments or drawing a circle with diameter AB).

The necessity of devising an explanation may be even stronger if pupils are asked to produce phenomena which are impossible to display visually. Questions like: *Is it possible to construct a triangle with two perpendicular angle bisectors?* (Abrougui [1]), or: *Is it possible to construct a triangle with two obtuse angles?* (Bergue [4]) require a proof, because it is not possible to obtain a triangle satisfying the required constraints on the screen. At the beginning of the task pupils were convinced of the possibility of obtaining such a triangle and they tried hard to draw one. But after several unsuccessful trials they became aware that it might be impossible and the question "why?" was then meaningful for them because it was based on their own experience.

Tasks (iii)

These tasks involve both the interpretation of Cabri-drawings in geometrical terms and their reconstruction by means of the geometrical primitives of the software. We call these *"black-box"* tasks. A Cabri-drawing is given to the pupils; they do not know how it was constructed and the facilities of the software giving access to the construction process are removed. The task for the pupils is to reconstruct the same Cabri-drawing, i.e. a drawing on the screen which behaves in the same way as the given Cabri-drawing does when it is dragged. These black-box situations cannot be given in a paper-and-pencil environment since a drawing does not convey information on its own about respective relationships among its parts. A discursive description of the figure should be given in addition to the drawing.

An example of such a black-box experiment in a grade 9 class is presented below (Fig. 4) (Boury [7]). The black box is made of a parallelogram ABCD and a central figure EFGH. The pupils must be able to construct a similar central figure attached to another parallelogram PQRS so that the whole has the same behaviour in the drag mode.

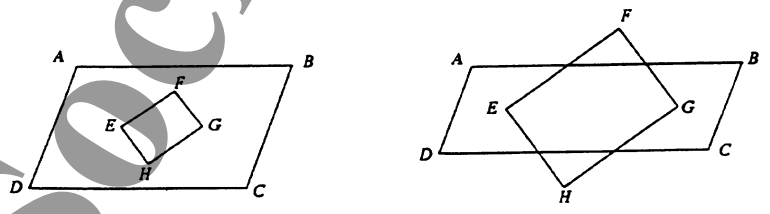


Figure 4

Some snapshots of the aspects of the Cabri-drawing to be reproduced

This kind of situation can also be used for learning about transformations. Two geometrical figures, one being the image of the other by an

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unknown transformation, are given as Cabri-drawings which can be moved; the task of the pupils is to find the hidden geometrical transformation, to reconstruct the image of the first figure and to check whether they then obtain the second one. Such black-box situations require several operations:

- the exploration of the unknown figure and its behaviour when it is dragged, an analysis of the geometrical properties which remain invariant by dragging, and in particular the distinction between what is contingent and what is necessary in the visual appearance of the Cabri-drawing under the drag mode;
- checking whether the supposed geometrical properties of the figure are satisfied by using other facilities of the software. For example, if three points seem to be on a same straight line, the pupils can draw a straight line passing through two of them and check whether this line goes through the third point when the drawing is dragged.

These two operations are based on an interaction between visualization and conceptualization, the first operation being an interpretation of visual aspects by means of theoretical notions, the second one consisting of validating theoretical interpretations by checking if two drawings remain visually superimposed while the drawing is dragged. This kind of task seems to contribute, in a very appropriate way, to pupils' construction of relationships between spatial and visual properties on one side and geometrical knowledge on the other.

In summary, tasks (i) and (iii) require the *a priori* use of geometrical knowledge, and tasks (ii) seem to motivate the need for a proof.

5. Observed pupils' solving strategies

Let us briefly report about pupils' strategies in the problem of constructing a parallel line through reflection and/or point symmetry. We observed the occurrence of three strategies:

- purely visual strategies;
- combinatorial use of geometrical primitives without definite intention;
- geometrical strategies aimed at a definite result.

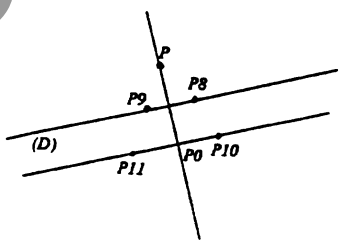


Figure 5

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In their use of geometrical primitives without definite intention, the pupils produced a parallel line but not passing through P. They observed that this line remained parallel to (D) even when P was dragged. For example, the line joining the points P10 and P11, obtained as symmetrical points of P around two points on the line (D), remains parallel to (D) (Figure 5).

What must be stressed is, that the features of the software played an important role in the move from one strategy to the next one:

- the drag mode disqualified the purely visual strategies;
- but it also enabled the pupils to notice a spatial invariant: the parallelism;
- because the pupils knew that the spatial phenomenon they produced by chance was the result of the use of geometrical primitives, they were sure that it could be reproduced using geometry.

The fact that the software "knows geometry" triggered the pupils' search for a geometrical solution. It is only at that moment that *they were really faced with a geometrical task*: how to reproduce this spatial invariant with the additional constraint that the line must pass through P? To manage this, they had to analyze their procedure; this analysis may lead them to understand why this procedure provides a parallel line. This actually happened with several students who were working in pairs.

It is important to note that all observed pairs except one finally solved the task with an appropriate geometrical strategy, but that this final strategy occurred after the pupils did other trials empirically or partially by eye. The software allowed through its feedback the evolution of the pupils. What is also important to note is that these pupils were already familiar with the software (about 20 hours of practice); however when faced with a new task posing them some problems they reverted to visual strategies although these strategies had shown their inefficiency.

6. The design of tasks: some conditions for fostering the use of geometry

A critical point of the solving process is the visual recognition of a geometrical invariant by the students, which allows them to move to geometry. Some spatial invariants are better recognized than others, and this ability depends of course on (i) the age and knowledge of the students, and (ii) the nature of the invariant.

A new aspect of the *didactical contract* (Brousseau [8]) due to the software must be stressed. It has been observed that the pupils usually try systematically every possible menu item on every object with the hope of discovering the solution or at least a hint. It is as if these menu items must be used because they are made available by the teacher. This last aspect

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must not be underestimated by teachers when they design tasks in a software environment. There should not be a high probability that a solution can be found through an empirical combination of menu-items.

The importance of the design of tasks and problem situations given to pupils is convincing when they can be observed evolving from a use of simple drawing to a use involving geometrical knowledge, from a "problématique" of the practice to a "problématique" of geometry (Berthelot & Salin [5]).

Section II. PROOF IN DYNAMIC GEOMETRY CONTEXTS

Celia Hoyles & Keith Jones[1]

Proof lies at the heart of mathematics, yet we know from research in mathematics education that proof is an elusive concept for many mathematics students. The question that we are now asking is whether the introduction of dynamic geometry systems will improve the situation – or will it make the transition from informal to formal proof in mathematics even harder? How far will innovative teaching approaches with computers assist pupils in developing a conceptual framework for proof and in appropriating proof as a means to illuminate geometrical ideas? Or will computer use be seen to replace any need for proof?

1. Proof in school curricula

Traditionally, in the school curriculum, proof has been taught largely in the context of Euclidean Geometry. It has tended to be presented as a formal confirmation of statements that pupils are told are true. This approach emphasizes the precise formulation of a standardized linear deductive presentation of argument, in which form is often perceived as more important than content. Research evidence suggests that such an approach to proof is fraught with conceptual difficulties for pupils. In the following, we summarize the main issues:

Pupils fail to appreciate the crucial distinction between empirical and deductive arguments and, in general, show a preference for the use of empirical argument over deductive reasoning (e.g. Williams, [49]; Balacheff, [3]; Martin & Harel, [40]; Porteous, [46]; Chazan, [9]; Finlow-Bates, [15]). For many pupils, deductive proof provides no more than evidence (Williams, [49]; Fischbein, [16]; Chazan, [9]). Proof is not 'used' as part of problem-solving and is widely regarded as an irrelevant, 'added-on' activity. Geometric proofs for example, and particularly the standard linear deductive style, frequently fail to 'connect' with learners who neither understand the purpose of the proof nor appreciate its role in mathematics

(1) The authors acknowledge with thanks the contribution of L. Healy & R. Noss to an earlier draft of this paper.

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activity. If formal proof is presented only as a way to demonstrate something that learners are already convinced is true, proof is likely to remain a meaningless activity (e.g. Hanna & Jahnke, [24]; Tall, [47]; de Villiers, [11]).

The challenge for mathematics educators is to find ways in which geometric proof has communicatory, exploratory and explanatory functions alongside those of justification and verification. There are considerable differences between countries in the treatment of proof in the mathematics curriculum and we must be wary of making generalizations based on evidence from just a few countries. Additionally we must always attempt to interpret pupils' responses from the basis of what they are taught, what is stressed in their curriculum and what is ignored. Clearly mathematical proof is unlikely to be appropriated without very careful teaching.

2. An approach to proof in the U.K. Mathematics Curriculum

In the U.K. a process-oriented approach to proof has been developed where pupils are encouraged to test and refine hypotheses in order to achieve personal conviction with little emphasis on generalisation and formal presentation of evidence of validity in the form of a proof. This approach to proving and proof is now enshrined in the Mathematics National Curriculum for England and Wales (see Chapter 6, Sect. I). There is a hierarchical sequence of progression whereby pupils first use inductive methods, move on to gain an appropriation of the difference between empirical evidence and mathematical argumentation, before finally (only for those who reach the highest levels of attainment) constructing formal proofs. These changes towards a process-approach to proof are matched by a reduction in the study of geometry to small fragments of shape and space.

The disappearance of geometry is perhaps not surprising given the present emphasis in U.K. on pupils generating their own empirical evidence which, in Euclidean geometry with ruler and compass, would be difficult to achieve. In effect this means that the majority of pupils have little chance to appreciate the importance of logical argument and few opportunities to prove a geometrical theorem. The implications of these curriculum changes on pupils' (age 15 years) understanding of the multifaceted nature of justifying and proving in school mathematics have been investigated during 1996 by Hoyles and Healy (Hoyles, [31]) in a nationwide survey of over 2500 high-attaining students. We can argue that the move towards a more process-oriented perspective represents an understandable attempt to move away from the meaningless routines that characterised geometrical proof in an earlier period. While some pupils managed to undertake the routines of Euclid correctly, few understood more about geometry as a result. But in trying to remedy this problem, we have removed two important aspects of mathematics: first that it is not sufficient just to see a pattern, it is necessary to understand it and to look at it scientifically. Second, it is not

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the single elements of geometry that are important, but the structures and relationships that bind them together. The question is: can the vital elements of proof and of geometry be salvaged without returning to the lifeless forms of Euclid which characterised the earlier mathematics curriculum?

3. The computer as a context for linking empirical and deductive reasoning

It is our conjecture that if we accept that formal proof should have a central role in the curriculum, we need to design and begin to evaluate innovative activities which enable pupils to make links between empirical and deductive reasoning throughout the mathematical activity. We need to find new contexts through which to introduce the use of clearly formulated statements and definitions and agreed procedures of deduction, which also allow opportunities for connections with empirical justifications. We suggest that the computer might offer just such a new context. This assertion must be treated with caution. We cannot assume that the introduction of the computer will bring about change, at least not change for the better. On the contrary, the challenge is to construct new pieces of learnable mathematics (based firmly on the old!) which are learnable precisely because they harness the potential of the new technology.

To date, work with computers in geometry education has largely been around the use of dynamic geometry software (e.g. Geometric Supposer, Cabri) where basic screen objects – points, lines and circles – can be created and explored through direct manipulation. They provide a model of Euclidean geometry which offers feedback through "dragging" as to whether constructions or theorems are correct. Parameters can be varied so that invariant relations are noticed, or lengths and angles measured so that 'results' are observed in the unchanging patterns of the measures. Thus pupils are able to generate ample empirical evidence of geometric theorems in ways which would have been difficult if not impossible before. Attention has tended not to be focussed on proving and proof but rather on the software's potential in aiding the transition from particular to general cases, since specific instances can be easily varied by direct manipulation or text-based commands and the text 'seen' on the computer screen (see for example Laborde & Laborde, [37]; di Sessa, Hoyles & Noss, [13]; Noss & Hoyles, [41]).

The use of a dynamic geometry package such as Cabri may provide an opportunity for some pupils to consider the "why..." in addition to the "what if..." and the "what if not...". Such an approach demands the provision of appropriate tasks, and in designing them there are two (connected) issues. The first issue involves ensuring that the pupils should experience the necessity of geometrical facts that are true in Euclidean geometry. The second concerns providing the pupils with suitable experiences to allow them to explain why these geometrical facts are necessarily true.

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In the following, we describe some work carried out with a lower secondary school mathematics class (pupils aged 12). The idea is to provide the pupils with opportunities to experience what could be called 'proof as explanation' (Hanna, [25]). The quality of the pupils' mathematical analysis suggests that the use of a dynamic geometry package such as Cabri, coupled with suitable tasks, may provide an opportunity for them to develop a basis for a fuller appreciation of the nature and purpose of mathematical proof.

In developing dynamic geometry contexts the following factors are critical:

- encouraging the learners to make conjectures focussing on the relationships between geometrical objects;
- providing the means for the learners to explain their actions and their results.

It is important here to distinguish between *drawing* and *figure*. Laborde ([35], p. 49) makes clear the distinction in the following way: *"drawing refers to the material entity while figure refers to a theoretical object"*. In terms of a dynamic geometry package, a drawing can be a juxtaposition of geometrical objects resembling closely the intended construction. In contrast, a figure additionally captures the relationships between the objects in such a way that the figure is invariant when any basic object used in the construction is dragged. The ability to check a construction by dragging appears to be a particularly important part of experiencing the necessity of relevant geometrical facts.

Healy *et al.* ([26]) suggest introducing pupils to the idea of "messing up" (or more accurately, not messing up). They define "messing up" in the following way: "After a figure was drawn it could be dragged to see if it became unrecognizable, that is whether the different objects within the design moved together in a sensible way or not" (*op. cit.*). While messing up was an idea that was easily appropriated, they found, after a study of how pupils understood the dependencies implicit in their constructions of some simple figures, that few had developed any appreciation of these. They did not, for example, realize why they could not change the relationship among constructed objects (see Hölzl *et al.* [30]). This study casts doubt on how far experimental work with Cabri can assist pupils in constructing systematic arguments – although much more work needs to be done to investigate this. Pupils need explicit mathematical goals if they are not going to use Cabri solely as a drawing tool rather than a tool for constructing geometrical figures.

A suitable context for work on "proof as explanation" is the analysis of static figures and their construction in the dynamic Cabri environment; in other words, asking the pupils to construct, given a geometrical figure drawn on paper, the corresponding geometric figure in Cabri such that the Cabri figure cannot be "messed up".

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An important consideration is to utilise some of the pupils' mathematical knowledge. A long-standing component of the primary geometry curriculum in many countries is the recognizing and sorting of various simple plane shapes. This entails such matters as identifying circles and various specific triangles and quadrilaterals. In the secondary curriculum, pupils are expected to know and use the properties of quadrilaterals and be able to classify them on the basis of their properties. Given these factors, and the fact that there has been both discussion about the classification of quadrilaterals (for example de Villiers, [12]) and research involving pupils' ability to classify them (for instance Fuys *et al.*, [18]), the 'family of quadrilaterals' provides a suitable choice of context for Cabri tasks.

In constructing quadrilaterals, a key idea is the use of the circle as a length-carrier. This, in turn, depends on the locus definition of the circle. Just as the idea of functional dependency may not be easy for lower secondary school pupils to grasp, neither might the locus definition of the circle be (or, at least, it contrasts with what is claimed by Bishop, [6] to be the notion of the circle commonly held by upper primary and lower secondary pupils: that is, that a circle is a disc). Some way is required to introduce the pupils to using the circle in the construction of other geometrical objects.

What might be a promising approach to this problem, illustrated by the work of one pair of pupils on a selection of these Cabri tasks, is described in the next section.

4. Children working on dynamic geometry contexts

The pupils reported on here are 12 years old, with no previous experience of using a dynamic geometry package although they had all used various drawing packages and other information technology resources. The class is an above-average mathematics class in a city comprehensive school whose results in mathematics at age 16 are at the national average. The mathematics teachers use a resource-based approach to teaching mathematics and the pupils usually work in pairs or small groups. The class has three 50-minute mathematics lessons per week.

Each of the classroom tasks requires the pupils, in pairs, to analyze a figure presented on paper. The pupils are asked to construct the figure using Cabri such that the figure is invariant when any basic point used in the construction is dragged. This means that the pupils have to focus on the relationships between the basic objects (points, lines and circles) necessary to construct the figure.

Pupils worked on a sequence of tasks, during many sessions which took place over a period of five months, with often several weeks between sessions, since pairs took it in turn to use the computer. During that time, of course, they continued with their regular mathematics programme. This included some work on certain aspects of 'shape and space' although none

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of it could be said to be directly relevant to the area of the mathematics curriculum that was the focus of the Cabri work. The tasks started with an initial exploratory session aimed at acquainting them with the software, progressed through a series of 'starter tasks' involving the construction of patterns of circles and lines so that they could not be "messed up", to a series of tasks which focussed on 'the family of quadrilaterals'.

The next paragraph presents some aspects of pupils' work in one of the 'quadrilaterals' tasks, in which pupils are given the figure of a rhombus and its diagonals on paper, and are asked to construct that figure with Cabri. When we analyze one pairs' work, we find that they tackle this with consummate ease, checking after each individual action that the figure does not mess up. Although the task refers to the shape as a rhombus, the pupils refer to it, in part of their explanation, as a diamond. This is what they say:

> The sides are all the same because, if the centre is in the right place, the sides are bound to be the same.
>
> The diagonals of the diamond cross in the middle, though they are of different sizes [lengths]. The diagonals bisect each other. The angles [where the diagonals cross] are the same. They are right angles.
>
> The opposite angles [of the rhombus] are the same. Two are more than 90° but less than 180°, and the others are less than 90° but more than 0°.
>
> This shape is a rhombus because the sides are the same, the diagonals bisect at right angles and the opposites have the same angles.

In order to write down their explanations, the two pupils needed appropriate support. The task on paper contained certain prompts. For instance, the task suggested that attention be paid to the sides, the angles, and the diagonals. The pupils are also prompted into focussing their attention on these attributes and are provided with technical language (such as bisect) when appropriate in order to assist the precision of their explanation.

The above provides some illustration of how the quality of the mathematical analysis of such geometrical figures, done by these particular pupils, improved significantly and observably within three sessions using Cabri. Nevertheless, it has to be stressed that this experience is founded on the following factors: a well-established and carefully-nurtured classroom culture that values mathematical thinking, a sequence of carefully selected tasks, and a range of appropriate prompts from the teacher (and possibly other classmates). It might be that, without this combination of factors, the outcome would have been very different.

In spite of the limitations of this study and the recognition that it took place in a rather particular context, we suggest that through working on tasks such as the one described above, the pupils gained some insight into

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the structure of plane geometry. They were able to explain the properties of a rhombus, and when they came to constructing a square they not only knew that the shape had to be a square but they also knew why.

In this process, both of experiencing the necessity of a result (in this case that the figure has to be a rhombus) and of coming to know why that particular result is obtained, the ability to check by dragging appears to be particularly important. This facility allows conjectures to be tested by focussing attention on the relationships between the geometrical objects that have been constructed. The commands *"delete object"* (and the object's dependents), together with *"undo"*, have also shown themselves to be useful in a similar way. The argument is, that experiencing the necessity of such geometrical facts involves perceiving (at some level) the axioms of the mathematical structure of plane geometry. Explaining why these geometrical facts are necessarily true involves constructing chains of reasoning. These are two of the essential components in a meaningful experience of proof.

From the evidence so far gathered, it may be that carefully chosen tasks using a dynamic geometry package provide a suitable context for some pupils to develop an appreciation of mathematical proof. In order to see how this view of proof as explanation develops further in the classroom, the process of acceptance within such a setting is a vital aspect. In due course, attention will need to be paid to that process and to how this contributes to developing in pupils the basis for a fuller appreciation of the nature and purpose of mathematical proof.

5. Discussion

The example serves to illustrate how it might be that in a curriculum that fosters exploratory activity, learners with dynamic software in a conjecturing atmosphere can develop a feel for proof as explanation, as to how certain 'inputs' lead to certain results. Yet explanation is only one aspect of proof (see de Villiers, [11]). What about the rather crucial aspect of proof as a test of validity? After work with this software is there any need for verification in a formal sense? Can 'a way to prove' be seen rather as part of the constructive process, or simply added to it as was the case in paper-and-pencil contexts? There is a danger that the use of this software will foster a process-oriented approach to geometry not possible before, but in so doing – at least in U.K. – rather than taking a step forward will simply replay the mistakes of the past, and limit the mathematical work of the majority to empirical argument and pattern-spotting.

Clearly, if proof is presented in the traditional way simply by replacing figures on paper with figures on the screen, we may expect to find little improvement in pupils' conceptions. But can we bridge the gap between induction and deduction by (for example) stressing explicitly the steps of construction, reflecting on these steps by the use of the 'history' tool of

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the software as a symbolic trace of the process, or stressing the use of the 'check property' function? This approach will also aim to develop an explicit characterisation of the problem in the computer context which if carefully chosen can provide 'the bones of a proof'. Pupils would conjecture about the relationships between geometrical objects, construct the objects for themselves, and prove the truth of their conjectures in various ways, not in a linear fashion, but one which is spiral and iterative.

In this scenario, construction and proof would be brought together in ways not possible without an appropriate technology. We envisage that, by constructing a 'generic' example with Cabri, pupils have to attend to the relationships they are setting up; and this can provide a basis for proving, together with a rationale for its necessity, provided that careful attention is paid to the primitive objects available, the activity structures promoted, and the pedagogy on offer. In this way, we trust we can develop a flexible appreciation of the roles of proof to include illumination, discovery and communication alongside those of verification and rigour.

Clearly an enormous amount of work has to be done, to choose a construction process that might lend itself to a formal proof as well as to investigate the construction/proving cycle systematically. This is being undertaken by the research study conducted by Hoyles and Healy during 1995 to 1998 (see Hoyles & Healy, [32]).

Every country has special challenges to face. In U.K. pupils tend to work in a conjecturing and experimental atmosphere which surely must be a prerequisite for productive work with software. Yet, there is already a noticeable trend to use powerful dynamic geometry tools in order to spot patterns, generate cases, measure lengths and angles simply to provide data. This data-driven approach could, if we are not careful, allow us to side-step all the important mathematical content which the geometrical domain is capable of offering. In particular the path to proof as explanation or verification is likely to be made still more problematic.

It is certain that other countries with different traditions and cultures will have other problems in trying to incorporate computer work into geometry teaching and learning. What is required from all the community is deep thought about the role of the tools in relation to geometical knowledge followed by careful and systematic evaluation of their use.

Section III. CAD TOOLS AND THE TEACHING OF GEOMETRY

Iman Osta

Computers and computerized tools are being increasingly used in many job activities. As they facilitate the realization of many tasks, and pro-

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vide a potential for more accuracy and neat presentation, they introduce new practices, and new relationships among the objects manipulated or processed.

Particularly, *Computer Assisted Design* (CAD) software is used in various workplaces where designing three-dimensional configurations is needed (decoration, architecture, civil engineering, robotics, industrial design, etc.). The use of this technology necessitates the development of new spatial capabilities, as well as an evolution of the actions and practices performed during the design process. Indeed, many aspects of the activity are taken in charge by the computer, but other competencies and skills need to be enhanced, such as adapting one's procedures to the functioning mode of the software, coordinating views of an object, interpreting the representations provided by the software, structuring one's perception of space in accordance to that implemented in the software, constructing mental representations of the reference systems underlying the CAD applications, etc.

1. Connecting geometry teaching to professional needs

If we agree that one of the educational goals is to provide youth with basic skills and competencies needed for later professions, what is the role of school geometry curricula in this respect? Wouldn't one of its important goals be to prepare pupils for this shift of emphasis imposed by the use of computerized tools?

From another point of view, how can mathematics educators benefit from the tools available, in order to enhance the teaching of geometry? It is legitimate to think that the new tools of work might provide new tools in the teaching/learning of 3D geometry, if properly used. They might especially help in developing spatial skills and overcoming problems of representation and visualization of 3D configurations in geometry classes.

All this calls for a revision of geometry curricula, in light of a dual relationship between school geometry and professional needs. In such a revision, the use of CAD software in the teaching/learning of geometry might be advocated, based on two reasons:

(i). Geometry taught at school is nowadays the backbone of many professions in which CAD software are used. This is one of the fields where geometry acquires a strong practical dimension. Thus, school has the duty of preparing pupils for that cultural and professional change, and to help them experience geometry with the new tools.

(ii). CAD software can be a very effective tool to overcome many problems in the teaching of geometry, especially problems related to visualizing, representing, interpreting, and transforming 3D configurations.

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2. Questions regarding the use of computer graphic tools

It is now known from research that computer environments used in teaching and learning situations, or in workplaces, change the nature of geometrical objects represented. They also change the kind of relationship the user has with these objects (Laborde & Laborde, [37]; Osta, [42]; Pea [45]). While working with computer software, pupils adapt their strategies to its constraints and functioning mode, and new meanings are generated by this continuous process of adaptation, thus constructing new knowledge (Osta, [43], [44]).

What are the main differences introduced by the use of various packages for graphic representations, concerning the nature of geometrical knowledge and the structure of space constructed? We think that it is important to conduct a micro-analysis to understand the delicate changes brought into the situation by the new graphic tools. Let's then take a closer look at a geometry learning situation integrating the use of CAD software.

The following analyses are bye-products of a research study (Osta, [43]) conducted a few years ago. A sequence of learning situations was designed so as to develop geometrical concepts and constructs related to 3D reference systems, through the use of a CAD computer package. The tasks involved constructing graphic representations of 3D configurations. The type of plane representation adopted is the "parallel projection" (known in French as *perspective cavalière*), mostly used in schools and geometry textbooks. Hereafter, we will use the expression "parallel perspective" to designate the drawing according to the "parallel projection", in order to distinguish it from the "orthogonal projections" used in technical drawing.

The sequence was experimented in a grade 9 class of a French school (pupils aged 14-15). Besides their regular geometry classes, these pupils had a course in technical drawing. They also had a limited prior experience with computers, but this experiment was their first experience with CAD software. The aim of the research was to analyze the mechanisms of interaction between pupils and the CAD software they were using while constructing various spatial configurations, and the way this interaction can mobilize and orient the mental construction by pupils of various 3D reference systems. The analysis showed:

- a concomitance between the detection by the pupils of the functioning mode of the software and the evolution of their geometric knowledge;
- a close relationship between the type of activity and objects involved, on one side, and the way space would be structured by pupils, on the other;
- a close relationship between the type of activity and objects involved, on one side, and the type of reference system constructed, on the other.

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3. Three different situations

The qualitative analysis of the teaching sequence above leads us to compare some situations of graphic representation when three different techniques are used: paper-and-pencil, technical drawing, and CAD software. This comparison would highlight the basic differences brought by the use of the computer, which underlines the necessity of a fundamental change in the geometry curricula including objectives, types of skills to be developed, and methods of teaching, in order to connect school geometry and spatial professional needs.

Paper-and-pencil situation

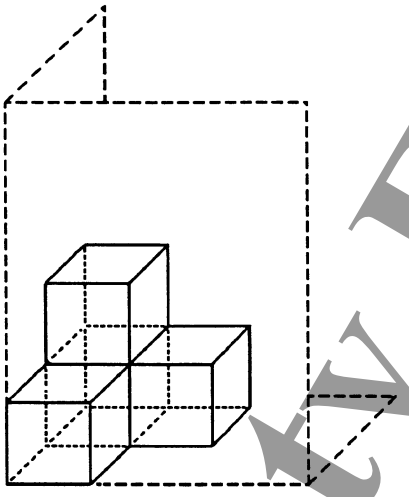


Figure 6

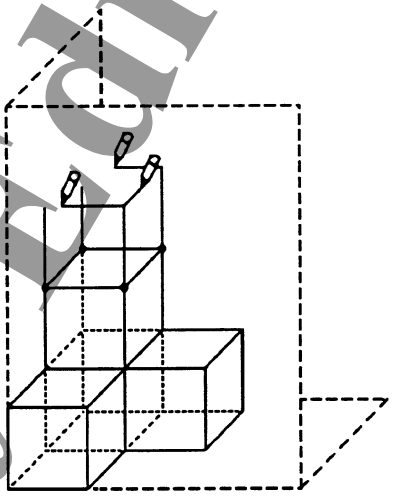


Figure 7

Starting from the configuration represented above (Fig. 6), composed of four identical cubes forming a trirectangular corner, the aim is to add a new cube on the top of the vertical pile of two cubes.

In a paper-and-pencil situation, the drawing activity should abide by the codes of the parallel perspective. The pen in Fig. 7 represents the various actions to be performed. In such a situation, the new cube can be drawn using the following steps:

- Extend the vertical sides of the cube on which the new cube is based (to be called base-cube).
- Take, on the four parallel lines obtained, equal measures from the vertices of the top face of the base-cube, each being equal to the side of the cube.
- Join each two consecutive extremities thus obtained, to draw the parallelogram representing the top of the new cube.

Considering those actions, we can notice the following:

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• The sketcher does not have to worry about the base of the new cube. It is implicitly considered as being the same facet as the top of the base-cube, hence already represented. No explicit action should be made in this respect.

• The only intra-object measure to be considered is the measure of the side. No relative positions of components or relative coordinates are needed to be determined or made explicit in order to complete the drawing.

• Drawing the new cube does not involve components of the already accomplished drawing, other than the vertices of the top face of the base-cube.

Technical drawing situation

In a technical drawing situation, a 3D configuration is represented by its orthogonal projections on the three basic planes of the space: the frontal, normal, and horizontal planes. Interpreting such a representation to reconstruct a mental image of the object necessitates a mental coordination of the three projections. Access to the parallel perspective of the object, or to some of its components, can only be done via the orthogonal projections. It is an indirect access, as compared to the direct access to the parallel perspective in a paper-and-pencil drawing situation.

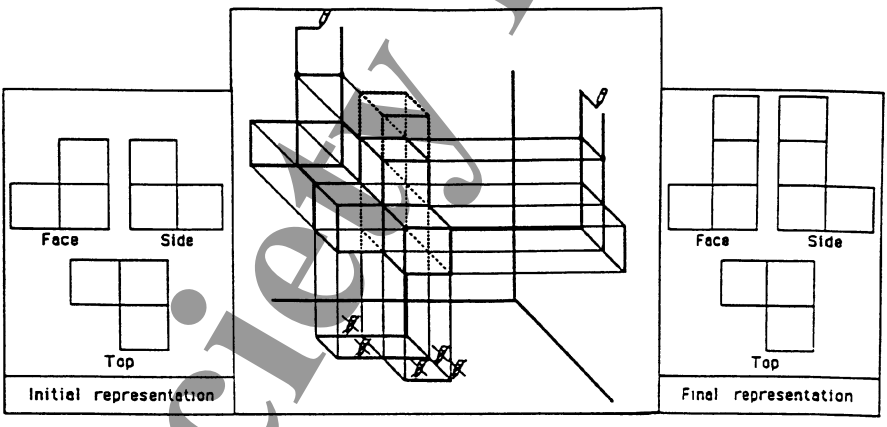


Figure 8

Practically, in the example of the configuration of the four cubes above, the base of the object, its top, and the base of the second-floor cube are represented by their projection on the horizontal plane (plane of the top view) which is parallel to them (Fig. 8). But some of these components have common projections on that plane. Thus, access to these different components having different levels can't be attained in the plane of the top view, but in the two other planes of projection: the normal plane (plane of the side view) and the frontal plane (plane of the face view), where the components mentioned have distinct projections.

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Suppose that, to the configuration represented, we want to add a new cube placed on top of the second-floor cube (which we have called base-cube). Since this new object superimposes, along the vertical direction, other identical objects, among which is the base-cube, it will have the same projection as those objects along this same direction (Fig. 8). Hence, no graphical action is to be performed on the top view, since the projection of the new cube on the horizontal plane along the vertical direction coincides with that of another already existing object. However the two projections of the cube on the frontal and normal planes have to be drawn. In Fig. 8, the sketches to the left and to the right are respectively the initial and the final states of the representation in technical drawing, while the pen represents the actions to be performed in the planes of the face view and the side view.

So, starting from the projections of the configuration on the three basic planes, the projection on the horizontal plane is kept as it is, and a square is added to each of the projections on the two other planes. As in the paper-and-pencil drawing situation, we can notice:

• Nothing new has to be drawn to represent the base of the new cube. The top of the base-cube, whose projections are already represented, would implicitly play this role.

• Drawing the projections of the new cube does not involve projections of components other than that of the base-cube.

CAD software

To analyze the basic differences introduced by the use of the CAD software, which may affect the type of geometrical skills involved in the teaching/learning situation, let us present some relevant aspects of the CAD software used in the research study, which might be considered as common features to most of the CAD computer tools available:

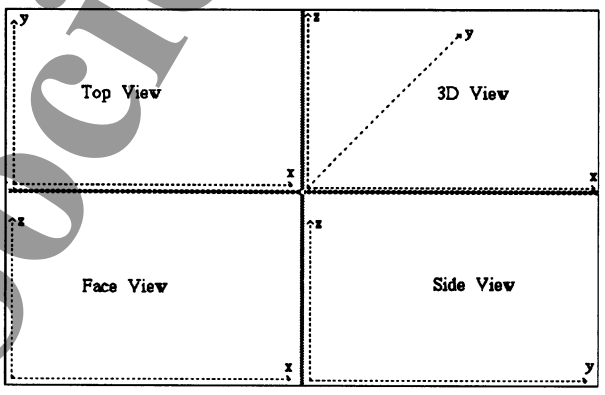


Figure 9

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The package considered is an interactive software used mainly to assist decorators, architects and designers in conceiving and constructing representations of 3D configurations. Each drawing screen includes four windows, three of which are the top, face, and side "views", the fourth being the 3D view (Fig. 9). The user is allowed access to the first three windows only. The fourth window is a control window where the user can see the various steps of the representation in parallel perspective of the configuration, with no possibility of accessing it or modifying directly its content.

Adding a cube in the CAD situation

Dealing with the CAD software is different from dealing with the system of views in technical drawing by the fact that, while in technical drawing we have orthogonal plane projections on three planes, the three windows in the CAD software don't display plane projections of the configuration, but actually the configuration itself, 'seen' according to observation directions respectively orthogonal to the basic plane of each window. Indeed, in the case of technical drawing, many components of the object, having the same projection, are treated as one by this projection. Their projections coincide in a plane surface, and the technical drawing logic does not require distinguishing them. It only requires coordinating them mentally. But according to the logic of the CAD software used (and many other CAD applications), a "view" in one of the windows may represent several components of the object, located at different levels according to the normal direction to the plane of the window, 'seen' as if they were 'crashed' on that surface. However, these components are distinguished within the software, and recognized as being distinct from each other, even if they have coinciding graphic representations in the window.

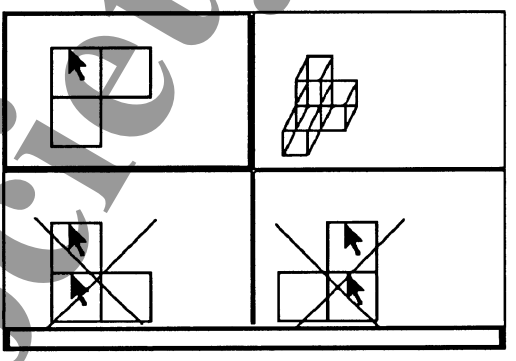


Figure 10

Going back to the example studied, let us analyze the operations through which one cube can be added to the pile of two cubes in the corner of the configuration. According to the functioning mode of the software, the user should select the icon "Prism". A dialog window is then displayed, "click an edge of the facet", requiring the user to select the facet which would play

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the role of base for the new cube. In response to that dialog box, the user should click one edge of that facet, in the window whose director plane is orthogonal to the prismatic surface to be constructed, namely the top view. In Fig. 10, the arrow indicates the action to be performed in order to select the "construction base" in the window of the top view, while the crossed arrows signify that no action is to be taken in the two other windows.

But three different faces are represented by the square whose edge is thus selected: the bases and the top faces of the two cubes forming the vertical pile. The user can have access to each of these many faces dynamically, through an alternating selection loop: as soon as the user clicks an edge, the software displays in dotted line one of the facets represented by the square having this edge, giving at the same time two options "selection OK" or "find other element" (Fig. 11). If the second option is chosen, another facet is displayed in dotted line. This process continues, alternating the facets, until the first option is chosen by the user. In such a case, the last facet which was put in dotted line is the one selected to be the base for the new cube (Fig. 11). A dialog box is then displayed, asking the user to enter the height of the prism to be constructed. Once the value of the height is entered, the new cube is drawn by the software (Fig. 12), as if it pops up from the designated base.

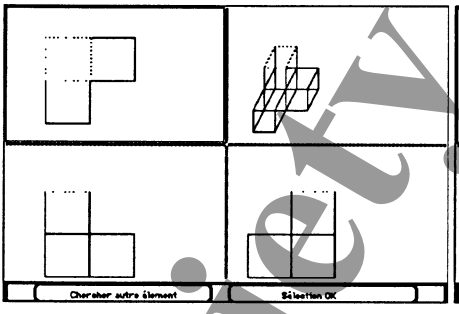


Figure 11

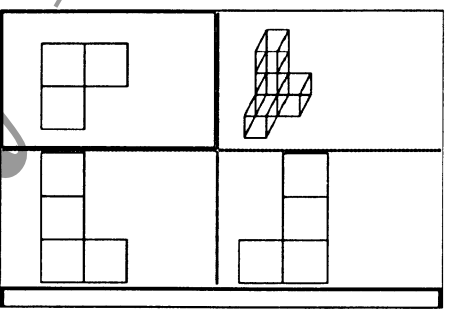


Figure 12

CAD situation versus technical drawing and paper-and-pencil situations

How is this CAD situation different from the previous two situations? What implications does it have on the nature of objects involved, as well as on the concepts and space structures eventually constructed by the pupils using it?

Contrary to the two preceding situations, drawing the various components of the figure cannot be done via points or segments (vertices or sides), it can only be done through facets. While in the paper-and-pencil situation, and in the technical drawing situation, the drawing activity involves mainly tracing segments in order to complete the skeleton of the graphic

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representation or its projections on the three basic planes, this activity involves constructing faces of the object using the software tools, or even a construction of the prismatic surface in one shot. This leads us to consider a special geometry which might be called a "facet geometry", in which the basic element is the facet (which might have any polygonal shape), instead of the point and segment.

Also, contrary to the other two situations, representing a prismatic surface can only be done in the window whose director plane is parallel to its base, hence perpendicular to the prismatic surface (the window of the top view in the example). This process is diametrically opposite to that of technical drawing, where the actions should be performed in the face and side views, and quite different from that of paper-and-pencil situation, where the actions should be performed directly on the drawing in parallel perspective.

We can go further in the analysis of the above difference, to see that the situations differ fundamentally in the way space is perceived and organized: in the paper-and-pencil drawing situation, the sketcher faces his drawing and the object to be drawn. In the technical drawing situation, even though theoretically the orthogonal projections on the three planes are drawn, research studies showed that the sketchers actually draw "views", which means that when they want to draw each one of the projections, they mentally observe the object along the corresponding direction, and draw what they 'see', thus they draw each time the "face view" of the object, observed along a different direction. In both cases the face view is privileged (Weill-Fassina & Rabardel [48]).

In the case of the software used, the procedure of prism construction provides a different perception of the space: in each of the windows (except the 3D window), space is considered as a combination of a privileged plane, which is the director plane of the window, with a privileged 1-dimensional direction, perpendicular to it. Depending on the window, and on the position of the configuration in the space, one of the three basic planes would be privileged.

We find also, in the process of selection of a facet among several facets having the same projection, a fundamental difference made by the CAD software, while teaching/learning 3D geometry: it allows a more dynamic and operative access to space as compared to the static and figurative access allowed in paper-and-pencil situations, or in classical technical drawing situations. This access preserves the nature of the objects and does not transform them into plane projections. It preserves the different levels of the components according to each of the three basic observation directions.

On the other hand, whereas in the two other situations, adding a representation of the new cube does not involve components of the configuration other than the top of the base-cube, within the CAD software the process of selection involves a family of components, namely all the facets having

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the same projection as the facet to be selected. This would make each step in the construction a new occasion for building relationships within the geometrical object, and making decisions about those relationships.

Even though this access process is more operative and more dynamic, it is far from being spontaneous or easily identifiable by the pupils or by novice users, especially when they were taught geometry in a non-computerized environment. From this point of view, the practices of paper-and-pencil drawing and the representations of space relevant to technical drawing may impose themselves and constitute obstacles to the particular mode of access and action in CAD situations. It is only through the use of these tools in problem-situations, and through adjusting one's practices and strategies to their logic, that these obstacles are overcome, while new knowledge is constructed.

4. From figurative processes to operative processes

To study more of the novelties brought about by CAD software to the teaching/learning of geometry, as compared particularly to paper-and-pencil graphic representation situations, let us consider the following example:

Given the graphic representation shown in Fig. 13 of two cubes having one vertical side in common, the aim is to draw a square identical to the faces of the cubes, representing a platform to fill the corner between the two tops of the two cubes.

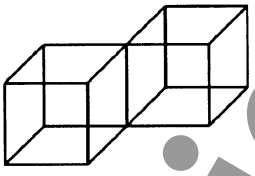


Figure 13

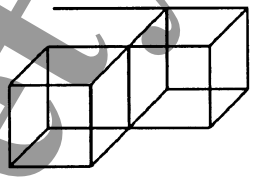


Figure 14

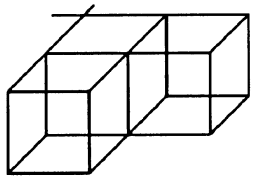


Figure 15

Drawing the facet in a paper-and-pencil situation

In a classical paper-and-pencil drawing situation, it is sufficient to extend the two segments representing the edges of the two cubes until they meet, as shown in Figures 14 and 15. No measures need to be made, no relative positions of components or relative coordinates need to be determined or made explicit in order to complete the drawing. Then, in this situation, the relative positions of the components can be controlled and determined, while directly acting on the drawing, by properties such as: connexity, prolongation, parallelism, etc, in transposing the problem from the 3-dimensional space of the object to the 2-dimensional graphic space. The sketcher's actions are then based on the direct correspondence between

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the object to be represented and the drawing to be obtained, via the parallel perspective codes and rules. Often, these codes remain implicit, the construction being guided by figurative processes and charged with effects of intuitive perception and cultural stereotypes.

Constructing the facet in the CAD situation

In the CAD situation, the 3D window is not accessible to the user, the sketch in 3D view being only a result provided by the software upon the user's actions on the three other windows. The user's role is to communicate to the computer the dimensional and positional characteristics of the object to be drawn, through its components, in each of the three windows representing the three basic directions of the space. The computer responds by giving progressively the graphic representation in parallel perspective in the 3D window, which serves only as a control window.

Assume the same graphic task above is to be performed with the CAD application considered. The construction process is bound by the structure of the space as implemented in the software, and by the reference system underlying it.

Reference system of the CAD software used

The space of the CAD software used is controlled by an implicit orthonormal reference system, composed of three virtual axes, mutually perpendicular: $\overrightarrow{Ox}$, $\overrightarrow{Oy}$, $\overrightarrow{Oz}$. It should be noted here that the axes don't actually appear in the interface of the software, they are implicit. In Fig. 9, they are drawn in dotted lines only to show in which way the space is organized according to this reference system in each of the four windows.

Space is considered, in each one of the windows, as a juxtaposition of a privileged direction, perpendicular to a privileged plane. It is isomorphic in each window to a non-associative product of three 1-dimensional spaces: $(\overrightarrow{Ox} \cdot \overrightarrow{Oy}) \cdot \overrightarrow{Oz}$ in the window of the top view, $\overrightarrow{Ox} \cdot (\overrightarrow{Oy} \cdot \overrightarrow{Oz})$ in that of the side view and $(\overrightarrow{Oz} \cdot \overrightarrow{Ox}) \cdot \overrightarrow{Oy}$ in that of the face view.

The user may have control of the coordinates as they appear in the interface through a communication window at the bottom of the screen, upon the selection of a graphical tool. The four values in that window are the three coordinates of the current point represented by the cursor, and the distance of that current point to the origin. These values change in real time as the cursor moves in the window (Figures 16-17).

One point of the space is characterized, in each window, by:

• Its coordinate along the privileged direction of this window (for instance the coordinate z if the point is to be considered in the window of the top view). Practically, this coordinate is assigned in a static way, by using the command *"3rd COORD."* (Figures 18-19). In default of such an operation, this coordinate is equal to zero.

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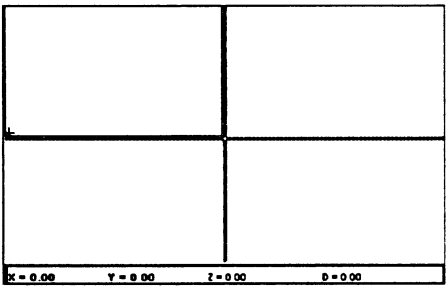


Figure 16

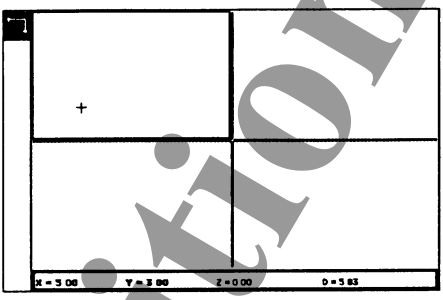


Figure 17

• The coordinates of its projected point on the privileged plane of this window. Practically, these coordinates are assigned in a dynamic way, by moving the cursor in the window and clicking when the desired position of the point or the desired values of the coordinates are displayed (Figures 16-17).

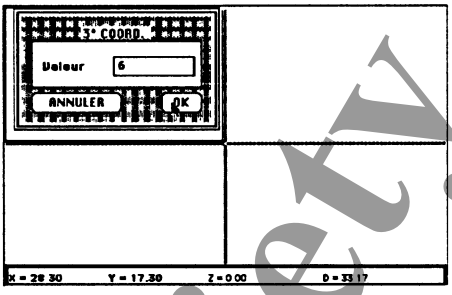


Figure 18

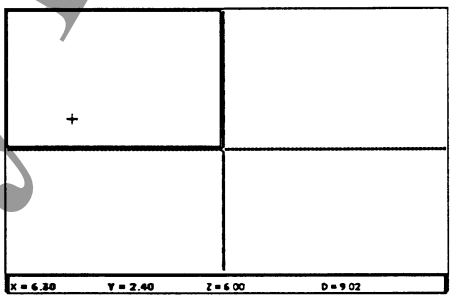


Figure 19

Once assigned, the 3rd coordinate with respect to one window would keep its value, and won't be affected by the displacement of the cursor in the window, as the two other coordinates are. Its value would change only when another value is assigned in the dialog box displayed upon selection of the command *"3rd COORD."* (Note, in Figs. 16-17, that the value of z remained 0 as the cursor moved, while in Figs. 18-19 the value of z changed from 0 to 6, which is the value assigned to it in the box "3*rd COORD.*").

Process of construction

We start from the initial figure of the two cubes having one vertical side in common, with their bases at the zero level with respect to the horizontal plane of the reference system (Fig. 20). The construction of the new facet should be done in the window of the top view. The function "rectangle"

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should be used, and the new square should be defined through clicking two opposite vertices. Drawing the square in this window of the top view can be controlled by coincidence of points and alignments, while the two squares representing the already constructed two cubes serve as controls (Figs. 21, 23, window of the top view).

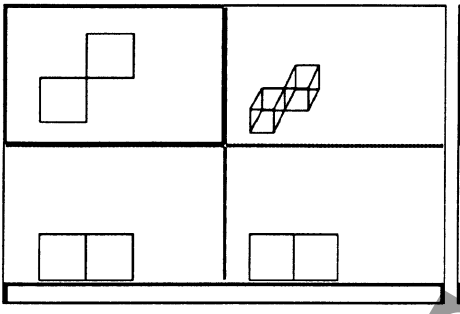


Figure 20

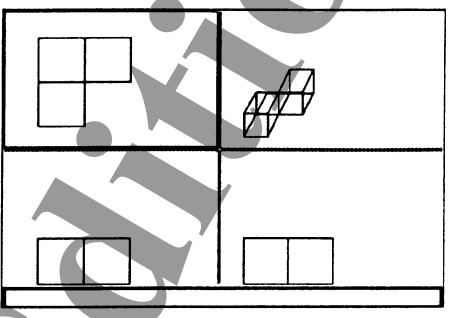


Figure 21

But before the construction of this square, the third coordinate should be entered, or else a result such as the one in Fig. 21 (3D window) will take place. Fig. 21 shows in fact attempts by many pupils to draw the facet in the top view without communicating the level of its plane. The facet is then drawn at the level zero, the same level as the bases of the two cubes. These attempts demonstrate a fixation around the technical drawing techniques which lead the pupils to consider the drawings in the three windows as plane projections, not as representations of the 3-dimensional object. It is only after several adjustments of their strategies to the functioning logic of the software, and after a long, time-consuming detection of the reference system underlying its space, that they could complete the task (Figs. 22, 23).

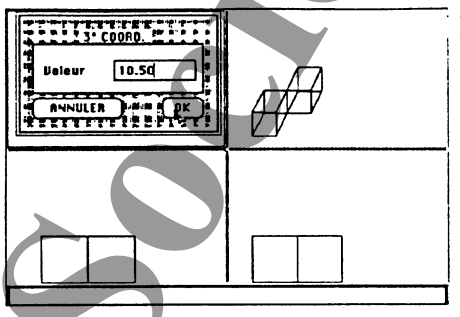


Figure 22

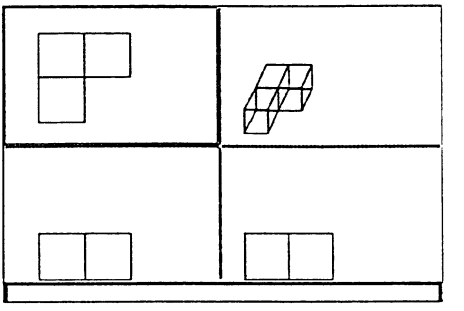


Figure 23

In such a situation, there is no way to draw the new facet without explicitly communicating to the computer information about its position relative

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to the other components, within the object itself. For the software used, the user should explicitly enter the coordinate of the plane of the facet with respect to the basic horizontal plane before drawing it. Such information pertains then to the 3-dimensional space of the object rather than to the graphic 2-dimensional space of the drawing.

This fundamental difference between the treatment in a paper-and-pencil situation and a CAD situation reveals the operative aspect (in Piaget's sense) of the geometric and representational activity in the latter, as compared to its figurative aspect in the former. The alignment and incidence relations in the graphic space of the drawing would not be enough to solve the problem. The value of the third coordinate should be calculated by the user in the space of the object, by making the spatial properties explicit and by relating them geometrically and analytically. If this process is so direct and easy in the case of the configuration of cubes considered here, because of the nature of the object, it would require more elaborate mental operations and geometrical knowledge and skills in more complex cases.

The research study (Osta, [43]) showed that this operative treatment is a main booster for learning about the geometrical objects involved, as well as about the reference system(s) controlling the software. The detection by pupils of the elements of this (these) reference system(s) is a necessary process for the realization of tasks within the CAD application. This process mobilizes a progressive mental construction of the reference system and of the geometrical relationships within it. This process is not possible in a paper-and-pencil situation, since many of the geometrical and numerical relationships remain implicit, and they are not always necessary for realizing the drawing tasks.

5. From one determined reference system to various possible reference systems

One of the most fundamental changes entailed by connecting school geometry to professional needs through the use of CAD software might be the need for a greater flexibility in the adoption of reference systems, as compared to one rigid conventional system usually used in traditional geometry classes. The latter system is often considered by teachers and pupils to be the only possible one, and the one which should be used in all situations, as if it had always existed, without being constructed by the pupils as a solution to specific geometrical problem situations.

Pupils should develop an ability to construct reference systems suitable to the situations they are tackling, to adjust to reference systems imposed by other situations and particularly by CAD software, and to shift from one system to another as needed. Indeed, while using CAD software, there could be many ways of structuring 3D space. The reference system controlling the software considered in this section provides an example where the axes

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don't have the same orientations as the usual conventional system (see and compare Fig. 25 using the usual classical system, and Fig. 9 where the reference system of the software is shown in 3D view).

Moreover, various structures and perceptions of a reference system might be incorporated in the same CAD environment, underlying different tasks or activities, depending on the procedure involved and on the strategy used to construct the components of a certain configuration. Particularly, in the CAD software used in this study one can identify at least three different structures of reference systems, each one being privileged by a certain category of tasks, geometrical objects involved, and practices used to construct them.

The first system

The first reference system is the closest to the conventional 3D reference system usually taught in school geometry. This system underlies the procedure of entering the three coordinates of a point through the dialog box displayed upon the selection of the tool "absolute coord." (Fig. 24). Theoretically, a point A of the space is defined by a triplet, formed by the algebraic measures of the orthogonal projections of vector $\overrightarrow{OA}$ on the three axes $\overrightarrow{Ox}$, $\overrightarrow{Oy}$, $\overrightarrow{Oz}$. The point A is then defined by the triplet: $(\overline{OA}_1, \overline{OA}_2, \overline{OA}_3)$ (Fig. 25).

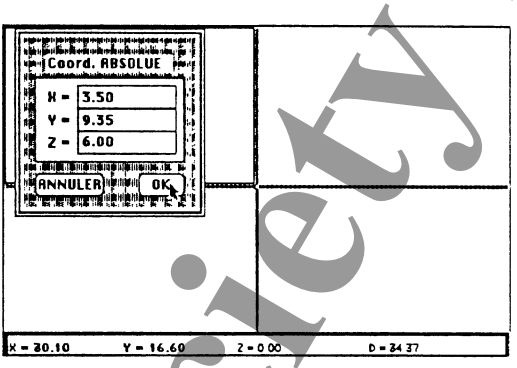


Figure 24

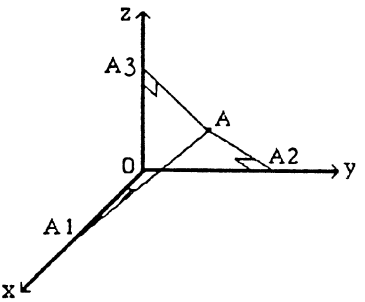


Figure 25

However, we think that such a reference system is far from being spontaneous or intuitive. Many of the software tools put more emphasis on two other perceptions and structures of 3D reference systems, each one being in fact a composition of a 2-dimensional with a 1-dimensionl reference system. They are described in the following.

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The second system

It underlies, for instance, the procedure "Prism", or the process of defining a point by constructing its projected point on a basic plane, then applying to it the "translation" tool, along a vector which is perpendicular to that plane. It consists of considering the projection of point A on one of the three basic planes (Fig. 26). The point A is then defined by a conjunction of two pieces of information: its coordinates in the projection plane, and its coordinate along the axis which is orthogonal to it.

In Fig. 26, the projection was made on the plane $(\overrightarrow{Ox}, \overrightarrow{Oy})$; the coordinates of point A are: (coordinates of $A', (\overline{OA_3})) = ((\overline{OA_1}, \overline{OA_2}), \overline{OA_3})$.

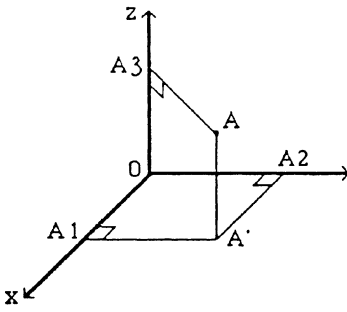


Figure 26

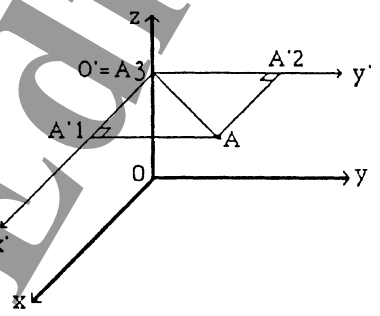


Figure 27

The third system

It underlies, for instance, the process of determining a plane of treatment, by using the tool *"3rd COORD."*, then defining dynamically its other two coordinates by moving the cursor in that plane. It consists of considering the projection of point A on one of the axes of the reference system (Fig. 27). Point A is then defined by the conjunction of two pieces of information: its coordinate $\overline{OA_3}$ along the axis on which it is projected, and its coordinates in the 2-dimensional system $(\overrightarrow{O'x'}, \overrightarrow{O'y'})$ obtained by translating the original system $(\overrightarrow{Ox}, \overrightarrow{Oy})$ along the vector $\overrightarrow{OA_3}$ (Fig. 27).

In Fig. 27, point A was projected on the axis $\overrightarrow{Oz}$; the coordinates of point A are: (coordinates of A in $(\overrightarrow{O'x'}, \overrightarrow{O'y'}), \overline{OA_3}) = ((\overline{O'A'_1}, \overline{O'A'_2}), \overline{OA_3})$.

Another important change is then related to new goals which become more important in the teaching of geometry, through greater stress on the formation of spatial sense, and the abilities of structuring space according to different structures, and of interpreting graphic representations. Less emphasis needs to be given to measuring and to figurative drawing skills, which are taken in charge by the software.

It is necessary that curriculum developers and teachers be aware of the changes in practices and in knowledge-models entailed by the use of CAD

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software, both as an end for being important in future professions, and as a tool for improving teaching.

Section IV. THE COMPUTER AS A CONTEXT FOR NEW POSSIBLE GEOMETRICAL ACTIVITIES

Klaus-Dieter Graf & Bernard R. Hodgson

We will present in this section some applications illustrating how the computer, when appropriately used, can serve as a catalyst providing opportunities for unprecedented pedagogical activities.

The framework we propose for our discussion concentrates on three different dimensions, the computer playing an important role in each of them: *mathematics, applications* and *didactics*. The dimension of mathematics, which involves the mathematical objects themselves underlying the teaching/learning situation, is concerned with the question of how the computer can help in the building of an appropriate understanding of fundamental mathematical ideas, in our case through the simulation of geometrical transformations. The dimension of applications shows the computer as a facilitating tool in constructing complex figures through repetition of basic mathematical procedures. Under the dimension of didactics, computer simulation is used to decompose some phenomenon so as to shed light on basic mathematical relationships and properties.

1. New applications of the computer in geometry: Yin-Yang-based tilings of the plane and kaleidoscopes

In order to motivate our discussion, we first present two concrete geometrical examples in which the computer plays a crucial and functional role.

Splitting and inverting the circle

Our first example is related to a solution proposed by L. Collatz to the problem of dividing, with ruler and compass, a disk into n regions of equal area. Fig. 30 shows Collatz's division for $n = 6$ as described in Step 1 in the following construction.

Figures 28 and 29 result from Collatz's division, followed by a "circular inversion":

Step 1: Starting from a circle with its diameter divided into n congruent segments, $(n-1)$ curves connecting the two endpoints of the diameter are drawn inside the circle, each curve being made of two semicircles located each in a different semi-disk and whose diameters are obtained by grouping

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the given segments into two adjacent segments. (The semicircles are drawn in such a way that all those contained in one half of the original disk have in common an endpoint of its diameter.)

Step 2: The curves are mapped outside the circle through a "circular inversion" type transformation.

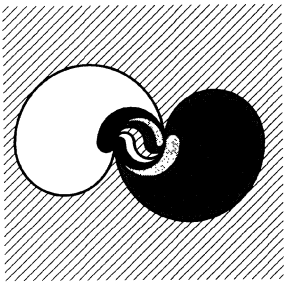


Figure 28

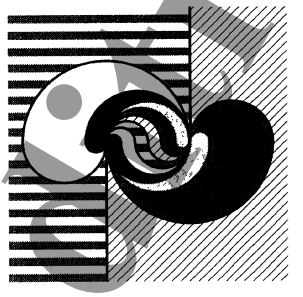


Figure 29

Since the entire plane thus gets divided into a certain number of regions, one can speak of a "tiling" of the plane; but it should be noted that some of the tiles are unbounded. Figures 28 and 29 give the resulting tilings when $n = 5$ and 6 respectively.

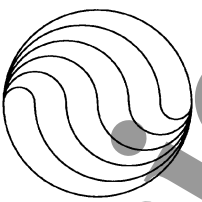


Figure 30

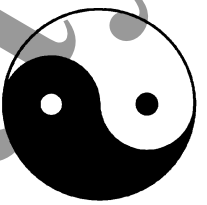


Figure 31

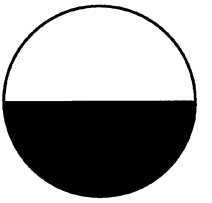


Figure 32

The specific case $n = 2$ has connection with the celebrated Yin-Yang or Tai Chi symbol of Daoist philosophy (Figure 31). In the following, we call the curve formed by the two point-symmetrical semicircles in the Yin-Yang symbol the *"yin-yang-* (or yy-) *diameter"*. The Yin-Yang symbol is famed for its aesthetic appeal, a striking aspect being its central symmetry. Fig. 32 presents another division of the circle into two symmetrical parts, yet less pleasant than the previous one. In comparison with the coolness of the straight section by a standard diameter, splitting the circle by a curved yy-diameter creates a greater sense of harmony. (We neglect here the two

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small inner circles of the Yin-Yang, although they play an important role in the Daoist conception of this symbol.)

In the next subsection, we pause briefly in order to present some basic relationships between circles and circular arcs.

On the measurement of circles

Geometry education in school stresses mainly the interactions of a circle with straight lines (diameters, chords, secants, tangents), the number π being a core concept in this study. One reason for this is certainly the importance of these interactions for the solution of practical problems related for instance to calculation of areas and circumferences. But a substantial study of circles can be carried out in a "π-free" context.

Historically, this has proved to be an important perspective, as is attested by the first proposition given by Archimedes[1] in his opuscule *Measurement of a Circle* (Heath [27], p. 91): *Any circle has the same area as the right-angled triangle whose two sides of the right angle are the radius and the circumference of the circle* (Fig. 33). This proposition is one of the "great theorems of mathematics" presented in (Dunham [14]). This proposition is discussed in the chapter devoted to Archimedes in the remarkable book of Serres on the history of science (Authier, [2]). The point is that Archimedes is establishing through this proposition a link between the two problems of *squaring* and *rectifying* the circle: the constant given by the ratio of the circumference of a circle to its diameter is involved in the expression of the area of the circle in terms of its diameter, which was already known to exist (see Proposition XII.2 of Euclid's *Elements*). This constant has been denoted π since the 17th century. This result, proved by Archimedes through his "method of exhaustion", is in direct relation to a preliminary one, stating that the area of a regular polygon is half the product of its apothem and its perimeter.

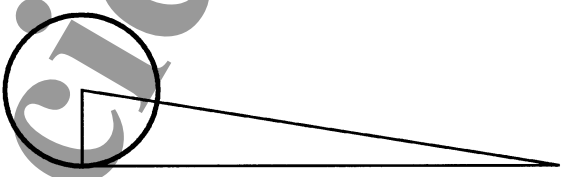


Figure 33

A more global point of view in school education would thus be, in addition to the study of the relationships of circles and straight lines, to examine as well the interactions of circular arcs and to find out mathematical relations and properties connected to such a context. For instance, let us denote

(1) The symposium on which this Study is based having taken place in Catania, close to Syracuse, it is only appropriate to pay tribute to Archimedes (287-212 B.C.).

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by r the length of the radius of the circle and by t the length of its yy-diameter. Since t equals half the circumference of the circle, one obtains the following formulae for the circumference and area of the circle: $C = 2t$ and $A = rt$. These formulae are not really 'practical', since t, unlike r, is not measurable with a metric ruler. But they do offer a different perspective on the relationships among the basic components of a circle, and they convey something of the harmony of the Yin-Yang splitting.

Correctness of Collatz's method

Let us now explain why Collatz's method connected to Step 1 above gives a partition (executable with ruler and compass) of the disk into n regions of equal areas. All the $n-1$ curves inside the circle (we call them *pseudo yy-diameters* for $n > 2$) have the same length t as the basic yy-diameter. Any pseudo yy-diameter thus divides the disk in two regions, each with a perimeter equal to the circumference of the circle. Moreover it is easily verified that these two regions have areas $r't$ and $(r-r')t$ respectively, where r' denotes the radius of one of the semicircles. We thus see that t (that is, πr in standard notation) is not just a term for half of the perimeter, but that it characterizes various circle-splitting sections composed of semicircles. A proof of the correctness of Collatz's construction follows from the preceding results. To show this, let us introduce the pseudo yy-diameter with smaller semicircle of radius kr/n (where $1 \leq k \leq (n/2)$ and consider the growth of one of the two regions it determines when we move to the next pseudo yy-diameter with radius $(k+1)r/n$. The added part, which corresponds to one of the n Collatz regions, is readily seen to be of area rt/n, thus independent of k.

Circular inversions and dynamic geometry software

Putting the Tai Chi symbol on the unit circle in the complex plane and applying the reciprocal complex mapping $z \to 1/z$, one extends the "yin" and the "yang" parts of the interior of the circle to the entire complex plane. Fig. 34 has been produced in such a way. Note that this geometrical transformation can be decomposed as a *circular inversion*(2) (i.e. the complex mapping $z \to 1/\bar{z}$) followed by a reflection in the x-axis (the latter being introduced for aesthetic reasons, so as to better separate the colored regions). Taking the image of Collatz' splittings of the unit circle and using colors to identify an inner region with its image outside the circle, one gets particularly fascinating coverings of the plane (Figures 28 and 29).

While it is possible to generate such figures by hand, the computer can be brought into play here in order to help us really 'see' what is going on, and also to facilitate our labor. It is particularly appealing to make use of the dynamic capacities of software such as Cabri.

(2) Given a circle of centre M and radius r, the *inverse* of any point P (distinct from M) is defined to be the point P' on the ray MP whose distance from M satisfies the equation $(MP)(MP') = r^2$.

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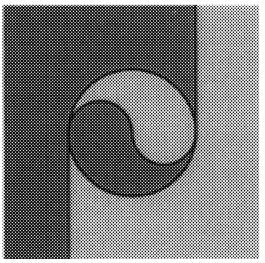


Figure 34

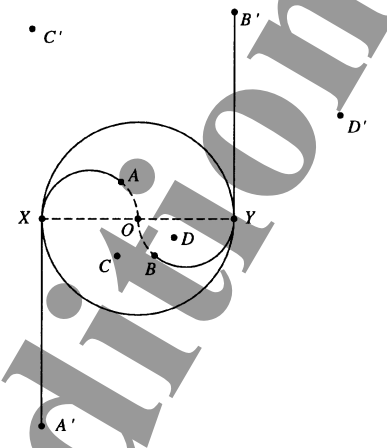


Figure 35

Figures 35 and 36 have been created through the animation capacity of Cabri. In Fig. 35, while point A moves along the yy-diameter from X to O, its symmetrical counterpart with respect to the centre, point B, moves along the same yy-diameter from Y to O. At the same time, points A' and B', their images by circular inversion plus horizontal reflection, travel along straight lines from X and Y respectively towards infinity. Points C and D illustrate how the inner regions get mapped. The pattern is similar in Fig. 36. (See Figures 28 and 29 for completed products.) Fig. 37 presents a variation on the idea of splitting and inverting the circle: the pseudo yy-diameters for n = 3 have been reproduced twice, a pair horizontally and a pair vertically. These four curves are then mapped outside the circle by inversion. A few corresponding regions of the resulting pattern have been indicated.

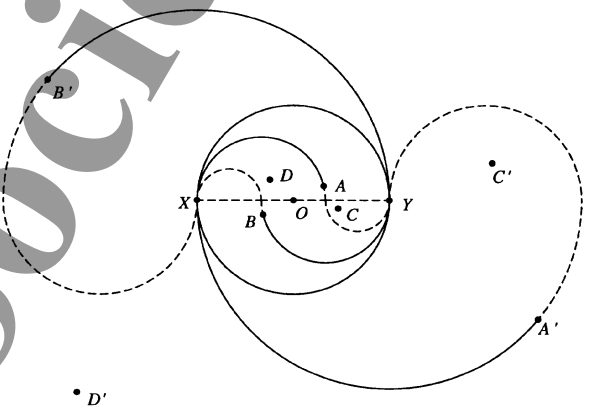


Figure 36

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It is clear that reproducing such figures on paper cannot convey the full flavour of what is happening on the computer screen. Seeing given points and their various counterparts actually moving on the screen, using when needed a color coding to stress the associations, is a truly amazing experience. In such a context, the transformations under consideration are not merely analytic objects described by a computational rule such as $z \to 1/z$, but they become true geometrical entities for which a strong visual intuition can be developed. An important feature of Cabri in this respect, which makes this situation basically different from an eventual paper-and-pencil one, is the fact that all the basic transformations of plane geometry, including circular inversion, are there as primitive operations, so the user does not need to mind about their analytic description or move to the complex plane in order to use them: they become geometrical objects readily available for exploration and having a full existence in the geometrical environment provided by Cabri.

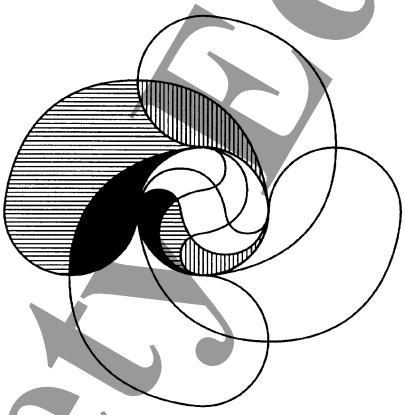


Figure 37

The kaleidoscopic effect

We now turn to our second example of a computer-enhanced geometrical setting. The features of circular inversion remind us in many respects of what happens, in a more elementary context, with the kaleidoscope. Both situations give rise to beautiful patterns generated by some basic geometrical transformations. In a kaleidoscope, the interplay of the two mirrors forming the device creates a rosace (or "rose-pattern") through the juxtaposition of the images of a basic motif placed in between these mirrors. Fig. 38 shows the pattern produced by two mirrors making an angle of 60°. The rules governing a "good" kaleidoscope, identified in the early 19th century by its inventor, Sir David Brewster, provide a good topic for paper-and-pencil exploration in an elementary setting (Hodgson [28]).

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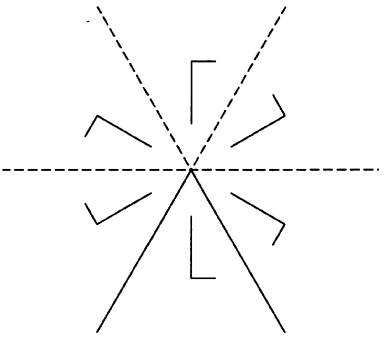


Figure 38

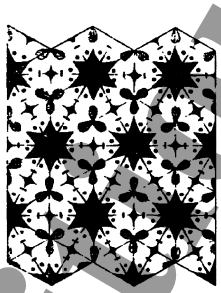


Figure 39

The kaleidoscope can also lead to tilings of the plane: the introduction of a third (or eventually fourth) mirror makes the rosace explode all over the plane, as illustrated in Fig. 39. Again Brewster had identified the principles of these "polycentral kaleidoscopes", for instance the fact that only three types of triangles are suitable: $(60°, 60°, 60°)$, $(90°, 60°, 30°)$ and $(90°, 45°, 45°)$. Any exploration of the kaleidoscopic effect must start from real kaleidoscopes, either commercial devices (some of very good quality can be bought at a reasonable price) or home-made ones. Only after such concrete manipulations is it appropriate to move to various stages that become more and more abstract and lead eventually to a formal description of the phenomenon (formulation and proof of theorems). A possible sequence of activities could thus run along the following lines:

- first, manipulations of mirrors in a free setting (instead of fixed mirrors in a tube), using as a motif small objects or a figure drawn on a sheet of paper;
- then, paper-and-pencil activities involving abstractly defined transformations, eventually performed with ruler and compass;
- finally, transfer of the phenomenon to the computer screen, making possible repetition of past experiments in a different manner and opening new opportunities.

Various software environments can be used in this respect. These will make possible automatization of tedious manipulations, so to facilitate construction of complex figures. Through an appropriate use of macro-constructions, they will also allow the user to concentrate on more global aspects of the phenomenon, non-pertinent details being hidden in the macro. But more importantly they will open opportunities which can only exist in a computer environment through manipulations which are physically impossible: for instance breaking a rosace in parts so as to understand its 'history', i.e. the way the images have grown together starting from the

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basic motif (the Cabri menu offers a "Replay construction" button allowing figures to be reconstructed in a step-by-step manner); or transforming this basic motif according to a rule for which there is no physical counterpart; for instance, a kaleidoscope resting on central instead of axial reflection, as discussed in (Graf & Hodgson, [21]).

In the next subsection we examine in more details the crucial role played by the computer in such contexts.

2. The computer in geometry education

We discuss the effect of the computer on each of the dimensions of the framework we presented in the introduction to this section.

The dimension of mathematics

In order to illustrate the mathematical dimension, we consider a specific topic of the geometry curriculum, namely transformational geometry. We select the following three transformations, which occur in school at different grades: reflection in a straight line (axial reflection), reflection in a point (central reflection) and inversion in a circle (circular inversion).

These are fundamental geometrical tools, which can be easily simulated with computer graphics. Moreover, when reflecting a geometrical figure, the computer allows a (nearly) simultaneous pointwise construction of the figure and its image (Figures 35, 36 and 42), thus making observable the transformation rule. While the first two transformations are rather standard in geometry education, the third one is not, possibly because of the more complicated mode of construction.

Axial reflection is most easily accessible by learners, since there exists a real model for it, the plane mirror. Central reflection does not have this real background, so it is considered to be more abstract by learners. Circular inversion again is related to reality via spherical mirrors (concave or convex), but this fact is usually not mentioned in the mathematics class, and left to optics in physics education. Very often mathematics students meet this inversion transformation only when studying mappings in the complex plane. A suggestive analogy with plane mirrors can be made for circular inversion: one can imagine a viewer inside a (hemi-) sphere whose inner surface is a mirror; the virtual images of objects standing inside the sphere, including the center, then expand outside the sphere, eventually going to virtual infinity.

We plead for an early integration of this transformation into geometry education, besides axial and central reflections, since it gives access to a wide variety of applications and methodologies. Again, the computer offers the crucial help in solving the technical and numerical problems related to the rather complicated construction of images under circular inversion.

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The dimension of applications

The dimension of applications in our framework concerns objects like kaleidoscopes and tilings. More precisely, we first consider computer simulations of real kaleidoscopes with two, three or four mirrors, based on a sequence of axial reflections. But there is no point in transferring such manual or mental activities to the computer in this way, unless this increases efficiency in learning. We believe that our examples are good illustrations of the pedagogical opportunities provided by the computer and of its scope-widening capacities.

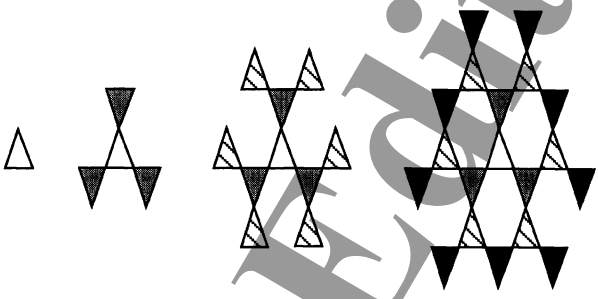


Figure 40

The computer first greatly facilitates exploration, allowing for a variety of investigations with little effort and making collection of data much easier. It also allows doing constructions which are impossible manually, like the pointwise (pseudo-) simultaneous construction of two or more figures, and makes possible the decomposition of a complex physical phenomenon, for instance the generation one at a time of the images produced by the interplay of mirrors, instead of having all of them happen at the same time.

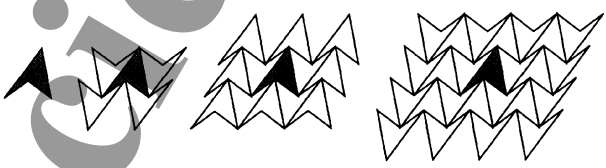


Figure 41

At a second level, the computer allows the study of devices which have no real material equivalent, for instance "fictitious kaleidoscopes" based on sequences of central reflections instead of the usual axial reflections (see Fig. 40). New mathematical patterns are thus created which have an interest of their own and which also lead to a better understanding of the usual phenomenon. Third, in the case of three or four "mirrors", both types of computer kaleidoscopes (with axial or central reflections) can generate

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tilings of the plane, depending on the relative position of the mirrors (that is, the shape of the triangle or quadrilateral they form). The striking result, visualized (not proved!) is that when working with fictitious kaleidoscopes reflecting in the midpoints of the sides of a tile, there is no restriction on the shape of this tile: any triangle or quadrilateral will do (see Fig. 41 for an example). Also, tilings involving two different types of tiles can result from fictitious kaleidoscopes. See (Graf & Hodgson [21]) for more discussions of such applications of kaleidoscopes, both real and fictitious. A formal proof that any triangle or quadrilateral can serve as a basic region, when repeating tiles by half-turns about the midpoints of their sides, can be found in (Coxeter [10], section 4.2).

A fourth application leads to some interesting ways of "tiling" the plane with figures bounded by circular arcs (a circle being the simplest example). Collatz's idea of splitting the circle into equal areas, as above, gives a tiling of the circle with each tile congruent to at most one other tile, all tiles having the same area and perimeter. Applying an inversion transformation, a tiling of the rest of the plane outside the circle occurs, which has a striking graphical appeal, especially if colors are used to relate a region and its image.

The dimension of didactics

The didactical dimension of our framework is marked by several methods. First, in the mathematics classroom, we can work on our mathematical objects and applications at several grades (e.g. 4, 8 and 12), considering enactive, iconic and symbolic phases in learning (Bruner): real mirrors and kaleidoscopes, geometrical modelling and symbolic reasoning. Second, through the use of computer graphics we offer efficient new modes for the inspection of mathematical processes, like transforming a figure pointwise or iterating a transformation (in a simulation of a kaleidoscope, for instance). Traditional actions with ruler and compass would be extremely inefficient for these purposes. Third, the knowledge of the existence of efficient tools from computer graphics also encourages pupils to attack constructions demanding a great number of elementary geometrical actions and to check conjectures by trying and experimenting.

We now illustrate the second method just mentioned by discussing a software package developed for geometry education in grades 7 and 8, the so-called "Symmetric Turtles", see (Graf [19]).

It is well known, and has been amply documented, that Logo's turtle graphics can help at the beginning of geometry education. A *running turtle* has been developed as a tool providing an extension of the ruler and compass. This follows the concept of Abelson's *dynaturtle*, but without inertia. To some extent the running turtle can be used like a pencil, being controlled with keys allowing straight-forward motions, right or left turns and marking points. Many figures of interest in plane geometry can thus be constructed.

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Besides this turtle, there are two more turtles, an *axial symmetric turtle* and a *central symmetric turtle*. They are controlled in the same way as the standard running turtle. However they not only draw the figure itself, but also its symmetric image, in a different color, in relation to the $x-$ or $y-$axis or the center point of the screen. This process happens simultaneously and pointwise (as was also the case while using the Cabri software for Yin-Yang tilings). This dynamic mode of construction for symmetric situations helps the user to recognize the properties of the mappings immediately and more easily than from a completely achieved figure. For instance, one can see a straight line remaining a straight line; or one can see the direction of the line changing under axial symmetry but remaining the same under central symmetry; one can also see, in the latter case, that the given line and its parallel image have different orientations.

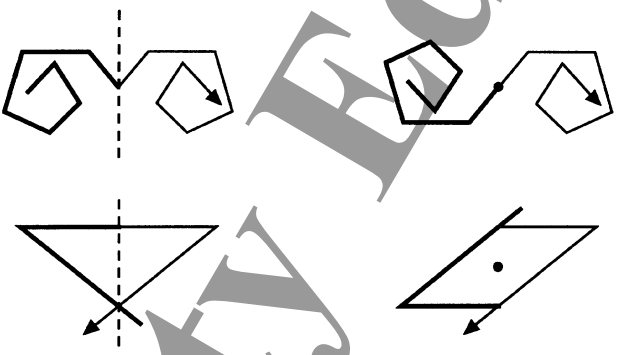


Figure 42

Fig. 42 gives some examples of figures generated with both types of symmetric turtles. Fig. 43 shows how the following question can be explored: What happens when reflecting a triangle in different positions relative to the axis or to the center of symmetry? More details about these tools can be found in (Graf [19]).

The idea of Symmetric Turtles can be generalised to study the properties of all the transformations of Felix Klein's "Erlanger Programm" from 1872, such as isometric transformations (translations, rotations, reflections), similarities, affine and projective transformations. This has been proposed and done by (Fischer [17]).

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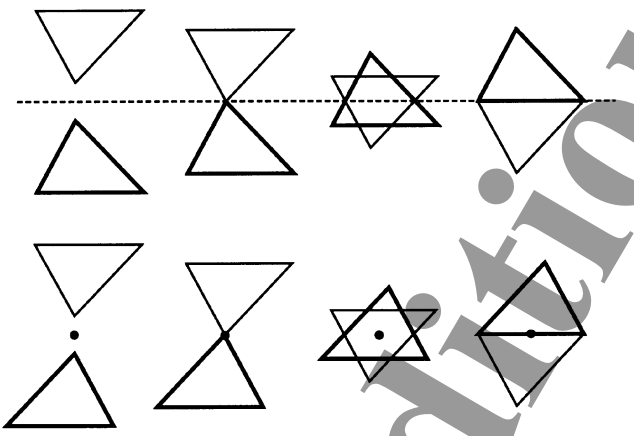


Figure 43

Concluding remarks

In spite of the considerable progress, during the last decades, in the development of computer hardware and software and of the many valuable educational experiments that have been carried out, computers are not so commonly used in schools today, including in the teaching of geometry, as one might have expected. A wide range of explanations can be put forward in connection with this situation: availability and power of machines, maintenance, complexity of user interface, pedagogical quality of the software, etc. Although all these facts are related to important issues, none are as critical as those pertaining to teachers' capacity and ability to cope with this new pedagogical environment. The learning and understanding of geometry by pupils in a technologically influenced environment places many expectations and demands on teachers, who must develop a wide range of competencies in mathematics itself, in informatics and in didactics. A more thorough discussion of this question can be found in (Hodgson [29]).

Some of the points raised in this section relate to essential aspects of general education. For instance, the Yin-Yang symbol, in addition to allowing intercultural connections, has an important interdisciplinary value: it is closely linked to philosophy and the patterns to which it leads are of interest from an artistic point of view, taking into account their aesthetic content.

In background of our work lies the conviction that complete mastering of a certain "micro-theory", such as the kaleidoscopic phenomenon or the splitting of a circle by arcs, can be for learners a truly positive experience with a long-lasting impact on their perception of what mathematics is really about and how they relate to it.

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Geometry in the 21st century – as has always been the case for that matter – will be a source of rich and illuminating situations with exceptional potential towards a better appreciation of mathematics. What is changing is that the computer now stands as a key ingredient in the teaching and learning of geometry, not only in attracting people through fascinating graphics, but also in allowing explorations otherwise too complex or even impossible to materialize.

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5. GEOMETRY IN THE CLASSROOM

Régine Douady & Bernard Parzysz

Presentation

When a teacher and pupils come together in a class, the institutional expectation is that the teacher is there in order to teach some knowledge, and that the pupils are there in order to learn this same knowledge.

But is knowledge really an issue for all the pupils who come to school?

The teacher's role, and often also his desire, is to communicate some knowledge to the pupils – in our case mathematics. But is that actually possible?

We would like the pupils to acquire a mathematical culture, and also ways of thinking and reasoning, knowledge and skills which enable them to deal with familiar situations as well as with unexpected ones. Is it possible to organize the relation between teacher and pupils, or even to make it evolve, so that mathematics become the actual object of the interaction, and that there is a chance for the aim to be reached?

This chapter consists of four sections.

Section 1 is devoted to a schematic analysis of the class situation according to whether mathematics is an issue for both the teacher and the pupils or just for the teacher, or just for the pupils, or for none.

Section 2 describes an approach with an example, starting from a geometrical problem satisfying some requirements, namely:

- The context is familiar to all. The question makes sense for everyone, whatever his knowledge is.
- Everyone is convinced that the problem has an answer. Everyone has an opinion and wishes to express it.
- But treatment requires some mathematical knowledge. Intuition is misleading. Then curiosity develops, observation disagrees with a-priori opinions, and mathematical work becomes a way to solve the suspense.

In sections 3 and 4 we describe a process of didactical engineering, i.e. a consistent organization of several lessons, on questions coming from a geometric problem depending on parameters. The teacher may fix specific values for the parameters, or leave them free to the pupils' choice. The decision depends upon the aim of the teaching, and on the expectation from the pupils.

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Choices may change according to the pupils' questioning, the questions they ask, the mistakes they make, their convictions, the difficulties they meet. In particular, the same situation may induce interesting mathematical work, accessible at very different school levels. This is the case for the problems we present in those sections.

1. The relationship between mathematics, teachers and pupils

1.1 Mathematical knowledge in the teacher-pupils relationship

It is appropriate to be concerned with the relation between what the teacher intends to teach, what he actually teaches, and what the pupils learn. One can question the best way to present a problem, a theorem..., how to predict the difficulties encountered by the pupils and how to help them to overcome those difficulties. But the words "to teach", "to learn", "to know" may themselves carry different meanings. And this will influence the choices and the expectations of the teacher.

However, all these concerns implicitly rely on the assumption that the pupils come to school in order to learn mathematics, that knowledge is an important or even essential object in the interplay between the teacher and the pupils.

Reality may be so, or it may be very different. Even though reality is always half-tone, one may consider four extreme cases, according to whether in a mathematics class the teacher and the pupils experience the same relationship towards mathematics or not. Writing T for the teacher and P for the pupils we can display the following matrix:

Interest/curiosity/passion for mathematics	T : yes	T : no
P : yes	**Paradise**	**Conflict** drift towards didactical fiction
P : no	**Conflict** but advantage for T	**Didactical fiction**

In the same classroom, it may occur that some pupils come to school in order to acquire knowledge while others only try to pass from class to class and go on as far as possible in order to get a good job, and yet others try to socialize and learn to be resourceful. They don't care whether mathematics or whatever else is taught there (Charlot & Bautier [5]). On the other hand, depending on the teacher's own personal history, mathematical knowledge, conceptions about learning, will to convince and imposed constraints, he will try to push his convictions forward or will just try to survive. Sometimes, even that is not too bad !

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So, teacher and pupils may be in phase or in opposition regarding mathematics, as the matrix indicates. Possible states are:

Paradise situation. This occurs in a class where teacher and pupils are passionate about mathematics. Geometry as a modelization domain or as a privileged place for reasoning may be a convenient field for the interplay. The work of the teacher is then to choose and organize the putting on stage, including the timing of knowledge so as to develop curiosity, in a way that has to be acceptable for the pupils and efficient in view of the learning aim.

Didactical fiction. This occurs when mathematical knowledge is an issue neither for the teacher nor for the pupils. In order for the teacher to do his job as a teacher and for the pupils to do their job as pupils, the class experiences a state of didactical fiction: the teacher shall teach something and the pupils shall learn something. They will be evaluated according to an explicit or implicit contract, internal to the class.

But where is the mathematics? What can the teacher do?

The usual procedure will soon be the following: the teacher proposes tasks to the pupils, which are broken down into elementary subtasks, algorithmized according to the pupils' needs, until an acceptable percentage of the pupils answer in an acceptable way.

The consequence of such a choice is, that the meaning of mathematical activity is given up. The 'magical' aspect takes over the rational one.

In order to avoid this situation, the pupils should have ways to control their production, other than doing the work again in the same terms. For instance, they could express the same problem in various ways and test consistency of information collected. They may also accept a result for a time and derive consequences referring to familiar situations, and again confront the conclusions with what is known in that subject ... Control would have to be time-economical and efficient in terms of certainty. Such a work would favour reorganization of knowledge and structuring of the memory. But such an intellectual and technical working of the pupils requires that they be able to choose, activate and relate pieces of knowledge coming from different domains. Moreover this form of scientific control under the responsibility of the pupils would have to be legitimate in the class, i.e. recognized and expected by the teacher. It should be part of the habits of the class, and clearly distinguished from the unavoidable evaluation of the pupils by the teacher.

In the 'paradise' state, mathematics is the heart of the didactical relation. But here (T no, P no) it is not so: the tendency will be to appeal more and more to memory, with little opportunity for organizing it. Pupils understand less and less why they are obliged to do mathematics. The teacher is led to divide the tasks and algorithmize more and more. But the course can proceed and go on. If the evaluation tests are chosen suitably – small questions in conformity with the habits – enough pupils will have

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good enough grades to pass. So survival is ensured in the short range for the pupils and the teacher.

But what in the long range for pupils who accept such a game?

What will be the fate for those who refuse it, or for those who fail in spite of their good will?

Conflict. This occurs when mathematical knowledge is a real issue for the teacher but not for the pupils. The issue for the teacher then becomes: to obtain a change in the relation of his pupils towards mathematics. That may be a big challenge for the teacher, who engages in a process of changing the relationships of pupils to school, the relationships between teacher and pupils and those among pupils. This requires the pupils to be able to enter a scientific activity and to be convinced that it is worthwhile.

It is the teacher's responsibility to convince his pupils that mathematics is alive as a science, that creation takes place everyday and that they themselves could be in a position to participate in it, either directly or by working in one of the many domains in the economic, technological or social life where mathematics is involved in an often hidden but essential way.

A change in the relation towards mathematics requires from the pupils that this discipline takes stronger meaning and that some tools are available under their control. This means that the teacher puts the pupils in position to have to make choices, to test their consequences, to control, discuss and possibly reconsider these choices ... The teacher has to make sure that pupils actually have the necessary cultural and technical knowledge.

This also means, concerning the didactical contract, that pupils accept being involved as actors, and don't confine themselves to the task of carrying out orders. It is in this learning context that the "devolution game" (Brousseau [4]) is unavoidable for the teacher.

Finally, this requires that the teacher puts the pupils in a position to establish strong relations between the building of meaning and a consistent and institutionalized recomposition of the pieces of knowledge concerned.

But what is his room for manoeuvre and what can he act upon, as he is submitted to institutional constraints and various forms of pressure?

In the classical school contract, knowledge and authority are on the side of the teacher. His task is to transfer a part of his knowledge, referring to the personal authority induced by his institutional position. If he wants the pupils to work under scientific control and not just under the threat of the teachers' evaluation, he will have to use his authority to help the pupils – in refusal to school and mathematics – to make a transfer of interest and curiosity towards mathematics and thus towards school and study (R. Douady [12] for an example of such a transfer).

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1.2 Hypotheses

A first hypothesis:

a) The acquisition by the pupils of mathematical pieces of knowledge, that are available and adaptable to various situations under their initiative, passes through the building of meaning for the notions concerned and their structuration, and memorization is made easy in the process.

b) For that, building the meaning must be an explicit goal in the teacher's view.

Question: how to manage so that meaning is actually built?

A second hypothesis:

For this purpose, it is possible to find problem contexts related to domains outside or inside mathematics which are familiar to pupils, so they can tackle them with their ordinary knowledge (including some mathematical elements) but specific pieces of mathematical knowledge have to be dealt with appropriately ("tool dimension" of mathematics).

A control lever occurs if there is a conflict between information collected from different sources, and the resolution of the conflict can be obtained through mathematical reasoning (see Sect. 2, below).

We think that the teacher may find there a way for mathematics, as a domain of study and thinking, to become a major issue in his interplay with the pupils, in case it had not been so, previously.

A third hypothesis:

For most pupils, the structuration of their mathematical knowledge requires a specific and explicit intervention from their teacher.

A fourth hypothesis:

In order for the pupils to develop, in the course of their school studies, a mathematical knowledge in its double dimension as tool and object, it is necessary that a long term mathematical life takes place in the class, where tool and object aspects evolve dialectically. This goal should be a 'class contract'.

An important element in the interplay within the class is the socio-cognitive interaction. This is what the next section is about.

2. Socio-cognitive interactions in the solving of a spatial problem in high school

The Piagetian approach to cognitive phenomena stresses the necessity of "unbalancing phases", followed by rebalancings, when a given individual constructs his knowledge. For Bachelard [2], a new knowledge is constructed against a previous knowledge, which proved insufficient or ill-adapted. Some researches have shown that such processes can be favoured by situations implying social interaction between pupils. Such a case is produced by problem-situations leading to the expression of a socio-cognitive conflict, provided some conditions are fulfilled (Perret-Clermont [20]).

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We now examplify such a process in a situation regarding the teaching of space geometry at high school level.

2.1 Introduction

Our example (see 2.3 below) was part of a research about the relationship between high school pupils (grades 10-12) and space geometry "figures", undertaken by the group "Geometry in high school" belonging to the DIDIREM research team of Paris-7 University (Colmez [6], Parzysz [17]). This survey first studied the drawings ("figures") made by high-school pupils to represent spatial geometrical situations, and how they understood the diagrams used in space-geometry courses. A first result was that the Piagetian classification in three successive – and hierarchic – stages (Piaget & Inhelder [21]) did not seem accurate to describe an individual's relation with the representation of space. An alternative model was then proposed (the knowing/seeing model), consisting of two antagonistic poles, called "seeing" (perception) and "knowing" (knowledge), between which a balance has to be found to represent a spatial situation. This balance depends on several factors, among which are the subject of the representation, its purpose, the – supposed or actual – receiver and the 'technical' competence of the individual (Parzysz [16], [18]). A consequence is that communication through drawings, figures or diagrams, like any other kind of communication, implies a connivance between the two protagonists involved, based in this case on a common 'culture' about representations. This is not in contradiction with the fact that young children are able to realize 'conventional' representations of a cube, since these are mere reproductions of a stereotype, which do not imply a mastery of the principles of representation (Dolle [8]). These results were confirmed by a subsequent research involving pupils from grade 3 to grade 10 (Colmez & Parzysz [7]).

Parallel to this, a study of the diagrams found in books dealing with space geometry over three centuries (from the beginning of the 17th century to the end of the 19th) showed that the knowing/seeing model applied to them as well, and that the current tradition in the representation of spatial configurations in geometry resulted from a compromise solution between "seeing" and "knowing", and not from an awareness of its geometrical status.

Starting from the results of that part of the study, a teaching of space geometry at grades 10 and 11 was then conceived and implemented, being intended to encompass an explicit teaching of some rules of drawing together with the teaching of the usual rules of space geometry and – of course – the developing of spatial abilities among pupils.

Such a 'global' conception of a teaching was motivated by three main reasons:

• *Firstly*, because the various aspects of learning mentioned above (space, geometry and representations) are connected, and interact with one another (Osta [15]).

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• *Secondly*, because the various types of representation traditionally used in the teaching of space geometry: bird's-eye views, front, above and profile views..., all belong – at least in a formal way – to parallel (or cylindrical) perspective, even if this statement can hardly be found in textbooks.

• *Thirdly*, because little is said about how to construct and interpret such representations; at best, some indications are given as "drawing conventions", although most of them are in fact properties of parallel projections.

A result of this situation is, that representations are generally used in a poor way by teachers as well as by pupils, although they could be of much greater use in problem-solving, as well as in the construction of geometrical concepts, and could even be a source of geometrical problems.

Another specificity of this research was the constant use of 3-dimensional models. A basic idea was that in space geometry, contrary to plane geometry, there is a wide gap between the situation and its representation by diagrams; thus, models can reduce this gap, because:

– they are an abstraction of the 'real' problem studied;

– nevertheless, they are, in some way, 'isomorphic' with the physical situation.

Moreover, although this situation often belongs to macro-space (i.e. involves 'big' objects), a 3D model of it belongs to micro-space, and thus gives a possibility of choosing various points of view, which is not always possible with the initial configuration. For instance, if pupils have to study a situation involving their own classroom about a problem requiring effective constructions and measures, they can first realize, then use a model of the room that will make the study of the problem (constructions, measures) easier.

2.2 The theoretical frame

In order to bring these ideas into reality, it was necessary to find some problem situations that fulfilled the above conditions, and which enabled the pupils to construct a good relation with the relevant knowledge. Another major idea was to favour interactions between pupils, in order to cause personal conceptions to arise and to generate open discussions about them, leading to the formulation of conjectures and to the working out of proof processes. These debates were expected to produce "action rules" and "theorems in act" (Vergnaud [23]) about: 1° space geometry, and 2° the representation of space, these rules being institutionalized later.

Some words must be said to clarify the theoretical frame in which this research was embedded. In a general way, this didactical engineering (Artigue [1]) was intended to make use of a "tool-object dialectic" (Douady [9]). This expression refers to a process in which a given piece of knowledge is first used as an implicit tool for solving a problem, then as an object of study, the properties of which are studied systematically; later on, this

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knowledge becomes susceptible to be used as an explicit tool for solving other problems.

Some conditions are required for the problem submitted to the pupils, in particular:

• the piece of knowledge aimed at must be 'the good tool' for the pupils to solve the problem;

• the pupils can have a start with a solution of the problem, using their existing knowledge (academic as well as non-academic).

Moreover, to be efficient, the implementation of such processes requires an appropriate management of the classroom, in which the respective roles of the teacher and of the pupils are well-defined and vary throughout time:

• in learning phases, the pupils must be given the possibility of making conjectures, to formulate and (try to) validate them through discussion;

• in teaching phases, the teacher is to the fore.

It must also be noted that the evaluation of the learning does not occur only after teaching phases, but starts from the learning phases, so long as the teacher pays attention to the arguments used by the pupils. Since these phases are based essentially on social interactions between pupils, the richness and quality of these exchanges are favourable conditions for the emergence of conceptions (correct as well as inadequate) and for their identification as such. Moreover, they allow the appropriation of knowledge by the pupils, as we shall see in the example discussed below.

2.3 The problem-situation

From a previous experience (Colmez [6]), we had come to the idea that the study of shadows cast on a plane was a suitable situation with regard both to space representation and geometry rules. It might have seemed easier to start with sun shadows, since this is a good concretisation of parallel projection; but a first experience had shown that some pupils had difficulty in admitting that sun rays falling on an object are parallel ("they diverge from the sun"). This obstacle led us to begin with the shadows cast by a lamp, which is, in fact, a central projection (even if we did not want to make it an object of teaching). Then we came up with the idea of the following device (see fig. 1): a telescopic pole is fixed on a board (0.6 m $\times$ 0.8 m),with an electric bulb at its upper end (power supplied by a small battery). A 'skeleton' cube (made of rods about 0,20 m long) lies on the board, its upper side painted red. The problem is to imagine what would be the shape of the shadow of the "red square" on the board, if darkness could be obtained in the class-room. This situation was chosen because we thought that it would make naïve conceptions emerge, leading to contradictions, of the same type as the one we had previously encountered: the fact that sun-rays diverge from the sun (which is true) apparently contradicts the parallelism of sun rays which can be observed daily.

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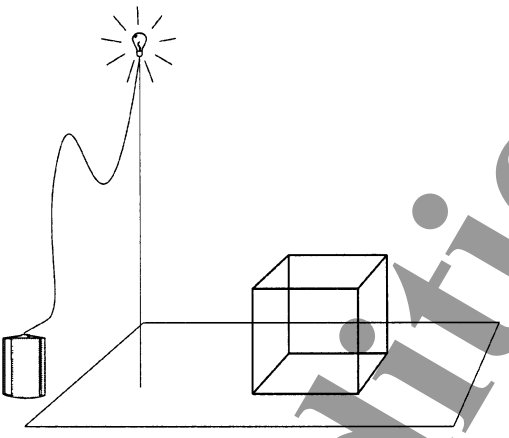


Figure 1

In the present case, we thought that the development of naïve conceptions is mostly based on the shadows of vertical objects (people, electric poles...), and that this fact would lead to contradictions if the pupils were asked to study the shadow of a horizontal object (the red square). From this point of view, the choice of the cube was motivated by two main reasons:

– this geometrical object is well known to the pupils;

– when the cube is lying on a horizontal plane, it comprises vertical edges as well as horizontal ones, and this we judged to be a favourable feature for the emergence of contradictions (see above).

2.4 The implementation (part 1)

Here we describe the progress of the sequence in a single case, although it has been experimented several times, with 10th and 11th graders. The various cases have shown that the situation is extremely 'steady', since the local conditions did not influence its general development.

First act: The device, lying on the teacher's desk, is first described to the pupils, who must then answer individually a questionnaire. By this, we wanted individual conceptions to be used and the results to be written down, enabling each pupil (and not only some 'leaders') to give his opinion, and constituting a support which:

(i) could be checked subsequently by the teacher, in order to supply an access to these conceptions, and:

(ii) would remind each pupil of his starting position about the problem, involving him more clearly and deeply in the subsequent debate.

The questions were the following:

– what is the shape of the shadow of the red square, when the cube lies in given positions (some being very particular, and others more general)?

– what happens to the size of this shadow when the pole is lengthened?

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– what happens to the size of this shadow when the cube is moved away from the pole?

When the questionnaire is completed, a collective counting of the answers takes place, which shows their great diversity; for instance, the correct answer for the shape ("*it is a square in all cases*") is given by less than one pupil out of ten (but, of course, nothing is said now about which is the good answer). The discovery of the multiplicity of answers puzzles the pupils ("*how can there be so many different opinions about so simple a situation?*"), and the problem – which was, at first, the teacher's own – becomes now their problem (this shift is known as the *devolution* of the problem to pupils).

Second act: A collective discussion can now begin, about the various answers given by the pupils to the same question. In this phase, the role of the teacher consists only in making the contradiction between these answers appear clearly, and in asking the pupils to explain how they came to their conclusions. Some positions are argued in a very convincing way by their authors, but nevertheless they remain contradictory: for instance, the shadow of the red square cannot be, at the same time, a trapezium ("because the rays diverge from the electric bulb") and a rectangle ("the right angles are preserved, because the square is parallel to the board")(1). According to Piaget's theory, we are here in an unbalancing phase, which has become explicit to everybody, thanks to the expression of different – and often contradictory – positions made possible by the collective discussion put into play.

There is indeed a way that could solve the problem once and for all: put the room in the dark and switch the bulb on; but this way is not available (says the teacher). And he asks: *How else could we know?* Now a phase of reflexion, individual, or, more often, with the neighbour, takes place.

N.B.: We also used this same problem-situation several times in training courses for in-service mathematics teachers. Their answers to the questionnaire are quite comparable with those of 10th graders; this phenomenon has been verified in various areas (mathematics, physics, biology, ...): when 'experts' are confronted with an unusual problem – even within their own domain – they most often react in much the same way as non-specialists. However, in the subsequent debate (but not in the questionnaire, except occasionally by one participant) the reference to homothety generally appears: the red square and its shadow are homothetic, with the electric bulb as the center. This settles the question, since all participants recognize the situation as one in which homothety is involved. But of course, such a conclusion cannot be obtained with 10th graders, because they do not have this knowledge at their disposal.

(1) Of course, mathematically speaking, a rectangle is a particular trapezium, but in the present context pupils use the word "trapezium" as opposed to the word "rectangle" (a trapezium having, in their sense, two parallel sides and two non parallel sides).

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2.5 The implementation (part 2)

Third act: Since a 'real' experimentation is not available, some pupils think of replacing it by a simulation, and they propose to use rulers or strings to materialize the light-rays; stretching them from the bulb and through a corner of the red square, we could know where they reach the board, i.e. where the shadow of the corner is. This is soon done by two pupils, coming to the desk to stretch elastic strings on the device (see Fig. 2), with the help of drawing pins (strings and pins are supplied by the teacher, who had anticipated the action). The idea of falling back on a simulation is always thought of by the pupils, because it is close to the real experiment (which, in their opinion, should be decisive) and stays on the experimental side. This experimentation shows a most surprising result in comparison with the answers to the questionnaire: the device is tilted to one side, so as to enable all the pupils to see the shape delimited by the four drawing-pins materializing the shadows of the corners, and everyone can see that "the shadow looks like a square".

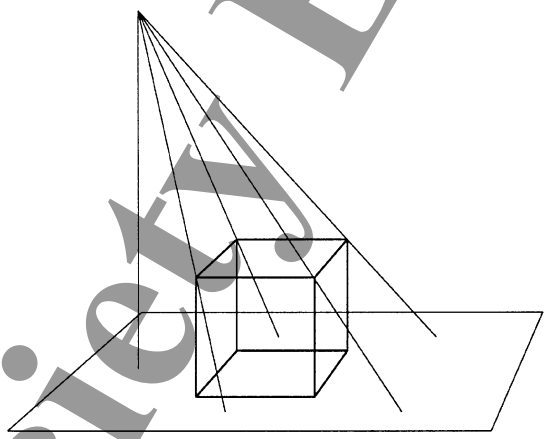


Figure 2

But the precision of the method (pragmatical proof) is not sufficient to be certain: some pupils do not agree and cling to their first position, so another kind of proof must be looked for. The teacher reformulates the question in another form: *Is the shadow of the red square really a square?* and asks for an argument which would convince everybody.

After another period of reflection, during which the pupils observe the device (with the elastic strings on) from their seats and discuss with their neighbour(s), there is once more a collective phase: the arguments given by the pupils are now on the geometrical side: their vocabulary is – at least partly – geometrical (line, plane, triangle ...), and they make use:

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• implicitly, of space geometry properties, which are in fact rules of action (two intersecting lines define a plane, the intersections of two parallel planes with another plane are parallel lines ...)

• explicitly, notions and properties from plane geometry that they already know (theorems on parallel lines, plane homothety, vectors) when they are considering a plane section.

They also come and draw figures on the blackboard, in order to show the others (who call for drawings) what they are talking about; these drawings are mostly plane subconfigurations of the spatial situation in play.

It must be noted that the starting of the proof processes is not easy, because the pupils do not see how to deal with space. Thus, the shift from space to plane ("change of framework" (Douady [9]) has often to be hinted by the teacher. But, once they get the idea, the pupils' arguments move easily from space to plane and back, using as tools in one case school knowledge (plane geometry and drawings) and in the other case non-academic knowledge, about space and its representations.

Finally, through several corrections and improvements suggested by pupils (the teacher's role is very restrained here), an intellectual path to proving the conjecture is collectively found, which seems to meet everybody's agreement (for more details, see (Parzysz [17], [18]). It is not a demonstration yet (in a mathematical sense), but a general idea has been brought out.

Fourth act: The teacher summarizes the proof process which has just been built, and emphasizes the points which were asserted without a (geometrical) justification. These points are of two types:

• in plane geometry pupils knew enough theorems, and a justification could have been given *within the theory*, but had not. The gap was easily filled without any difficulty;

• in space geometry some assertions were admitted without protest from the other pupils (for instance: "let us consider the plane of the two elastic strings", "the shadow of AB is parallel to AB because the red square is parallel to the board", etc.), but in this case there is no theory supporting such assertions (at least, not yet). Subsequently, the teacher institutionalizes such assertions, i.e. he decides that – together with some 'primitive' notions and definitions – these assertions, once transformed to fit in with these notions, will be part of what will now be called "space geometry" as "theorems". For instance, points, straight lines and planes will be primitive notions, and the following sentences (which are tranpositions of pupils' assertions quoted above) will be theorems:

(T1) *Two intersecting lines determine a plane.*

(T2) *The intersections of a plane with two parallel planes are parallel lines.*

By now, the rules of action have become genuine theorems, and can be used as such. They are noted down in the copy-books, and thus integrated

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into the 'memory of the class'. In this phase (institutionalization), the teacher is to the fore, since he points out which assertions will become theorems, and under which formulation. But it is important to negotiate this 'legal status' with the pupils, by showing them that in fact they used them in their own arguments – although in an informal way. Some of the new theorems (such as T1) will be admitted, and some others (such as T2) demonstrated, using both plane geometry and space geometry theorems, for it is important that the pupils see by themselves that these theorems are useful for activities; this is an essential part of their legitimation. But what has been solved by now is just one part of the initial problem (i.e. the shape of the shadow); the question of its size is still open (see the questionnaire above). In fact, this often appears as a bye-product of the earlier study, when proving that the shape of the shadow was a square involved figures in which the pole is represented (Fig. 3). When this is not the case, the teacher reminds the pupils of this question; this leads to another collective phase, very similar to the previous one, but much shorter, and using the new theorems as tools.

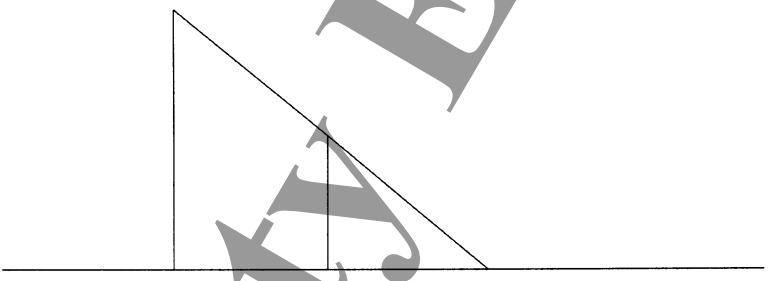


Figure 3

2.6 Nature and role of socio-cognitive interactions

The short sequence, just described, was included in a longer progression which extended over several months. This sequence was followed by others, intended to move from lamp-shadow to sun-shadow in order to deal with the question of drawings, as well as institutionalizing other definitions and theorems in space geometry (Parzysz [17]). But we will now take a special interest in the actors of the first situation, i.e. the teacher and the pupils.

In such a situation, the role of the teacher is obviously fundamental, firstly before the beginning of the sequence (since he chooses the problem-situation which will be 'lived' by the class), but also during its progress. At some times he acts as the leader of the group (posing the problem, institutionalizing the results worth remembering); he can also provoke a shift toward another mathematical field, or another register, which will provide the pupils with a starting point for a proof (in the present case: move from space to plane geometry). But at some other times he must try

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to be discrete and let the pupils express their own ideas: in this case, he acts as a moderator, stimulating the expression of ideas and being careful to allow anybody to speak (and not only some 'leaders'). It can be difficult, sometimes, to know what to say (or not to say), in order to summarize a discussion between pupils without introducing one's own ideas into it; this requires a great deal of patience and some 'mastery' of the situation. Let us notice that, in the present case, the teacher is helped by the device, since the simulation is a mediation providing an answer which – though partial – facilitates the formulation of conjectures by the pupils.

The teacher has also to insist on contradictions that appear between the positions of two pupils (for instance, the shape of the shadow being a rectangle is contradictory with its being a trapezium). The fact that the pupils become aware of such contradictions is a challenge for them, making the devolution of the problem easier. The teacher must also be watchful for specious arguments which sometimes are very convincing, causing thus a dangerous slip (Boudon [3]), and insist on them, in order to let the pupils become aware of their weakness or fallacy.

On the other hand, there are mainly two kinds of interactions involving pupils: with the teacher (answering or asking questions) and among themselves. The latter appears to be most important, as could be seen above. More specifically, the making of new conjectures comes in through discussions and new formulations of the problem. If there were no exchanges of ideas between the pupils, such a process would have no chance to be really fruitful. Actually, these discussions enable every pupil to confront his position with the others, and eventually to change his point of view (de-centration), to think of an argument for or against his own opinion or another's, or propose a new path.

Besides, the objection of a pupil, or his failure to understand another's position, forces the latter to bring an argument into play, in order to convince the other(s) (and himself as well); he will make use of various forms of proofs, some based on more or less formalized reasonings, some on visual evidence. The pupils may then become aware that, in a collective discussion, a given argument can convince somebody but not somebody else, and that a mathematical proof is a special kind of proof, following particular rules which are agreed by everybody. Moreover, such an argument will possibly reveal – implicitly – action rules and personal conceptions that will give the teacher insights on possible obstacles to the construction of concepts, and enable him to take them into account in his further teaching.

In conclusion, as can be seen from this, the success – or the failure – of such a didactical engineering depends mainly on two factors:

1. The choice of the situation, and its analysis prior to its implementation, from the points of view of the knowledge in play, the pupils' conceptions about this knowledge and a methodology which seems to be well suited.

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2. The management of the class, and the various didactical contracts which will actually take place through time.

The choice of a 'good' situation is important, but the role of the teacher, in his interactions with the pupils, appears essential, not only with respect to the evolution of the situation (this can be, to some extent, mastered by the choice of a situation which is not very sensitive to small disruptions), but also with respect to the relation of the pupils with the knowledge at stake. From this point of view, good management of the interactions with and between pupils is important, and ultimately affects the question of the reproducibility of didactical situations.

In the next section, we choose a set of problems and develop the *a-priori* analysis specifically.

3. Didactic engineering constructed on problems where algebra and geometry are mutually enriched

3.1 What is it about?

The task is to organize a series of mathematical working-sessions where the responsibility of the teacher alternates with that of the pupils. The key point in the teacher-mathematics-pupils relation is the working by the pupils on problems which are chosen by the teacher with a definite teaching intention. The problems are subject to requirements which will be stated later.

The engineering is arranged so as to take place in several phases, in order that:

• The pupils should be able to exploit their numerical and algebraic knowledge and skill. They should be able to use their geometric perception and their graphical practice. Moreover, they should be able to let these domains interact during their work.

• The teacher should be able to procede with his teaching objective, saving working space so as to integrate the unexpected ideas that may come from the pupils. He will thus be in a position to adapt the initial project to the new situation and even, if needed, to modify it completely. Moreover, the teacher should be able to evaluate what the pupils have actually learned, what is explicitly available under their control for work on new problems.

Such a didactical engineering should lead the pupils to develop their knowledge in the various domains involved in the problems. In particular, in geometry, *seeing* should evolve towards *knowing*.

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3.2 Choice of geometry and algebra

Both domains occupy an important place in all school curricula starting at the beginning of high-school, and their ground has already been visited, at least implicitly more or less consciously, starting at elementary school.

– *Geometry* allows bringing into play intimate convictions, coming from familiarity with the environment in which we live or with visual perception. This assistance lets us enter the problem with some ideas. But, at the same time, it can be misleading or take us to a dead end. It is then essential to have other tools at our disposal in order to allow the search to progress, to ensure the control of reasoning and guarantee the consistency or, at least, to trace inconsistencies, incompabilities, even if some questioning is left.

– *Algebra* can be a model-framework and can provide tools adapted for dealing with the problem. It may occur however that this holds from a mathematical point of view, but that the knowledge of the pupils in algebra turns out to be insufficient for the situation. In such a case, the appeal to numerical reasoning or the return to a geometric point of view, possibly different from the first one, may unjam the situation by suggesting a resort to other methods, such as graphs viewed geometrically as part of the Euclidean plane.

– *Meaning and forgetting meaning in algebra.* While reference to a context is invaluable when testing understanding, it is not sufficient. Successful handling of algebra comes from forgetting the contextualisation which gave rise to it, and requires a certain familiarization with technique.

So, we need to take account of the influence of meaning in the development of algorithms, and at the same time work towards a detachment from it. Thus, it is worthwhile to think of learning algebraic manipulation in terms of an equilibrium or interaction between the construction of meaning and a technical familiarity with algoritms.

3.3 Requirements for the problems

• *The mathematical situation should depend on at least one parameter.* For, the interest of problems depending on a parameter is manifold:

– From the mathematical point of view this provides a better understanding of a question by putting it in a wider background. In the problems below (see 3.5 and 3.8 below), what makes a rectangle exist or not? This gives meaning to the existence conditions and directs the solving process. Investigating several particular cases with various behaviours provides milestones in the range, and may lead to a study of all possible cases with their domain of occurence.

– From the teacher's point of view it enables him to deal with lack of homogeneity among the pupils. Changing the data may make the problem easier or harder in its conceptual and/or technical dimension. Each pupil deals with a particular problem, or a group of three or four pupils works

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on three or four problems in the field. Anyway, the whole class deals with a complete family of problems.

Working on a given family of problems, communication and exchange between pupils becomes possible in spite of the difference between the problems studied, the cognitive variation and the mathematical competence of the pupils. By favouring the circulation of the solving processes and debate on their validity or relevance, the teacher facilitates for each pupil the 'distanciation' from a specific situation or process. In this way, the process of decontextualisation and depersonalisation is kept operating.

– From the pupils' point of view, by confronting their personal choice and the consequences of the study, the conceptions are enabled to evolve: for instance from the notion of *constant* to that of *variable* or *parameter.* This enables the pupils to develop a reflective attitude toward their work, and thus to develop their autonomy.

The limit of these advantages lies in time-managing. This time-constraint also requires a second thought about the teaching-learning relation in the long run, on the year scale or more. It is conceivable that a question may be asked at a given moment in the year, partially solved according to the knowledge of the moment, and resumed several months later, when pupils are better equipped. This requires the teacher to take explicitly into account, in his decisions, the distinction which is essential in the scientific context as well as in that of teaching, between three attitudes on the pupils' side: to *understand* a question, to *ask* a question, to *deal with* a question.

• *The statement has a meaning for the pupils.* This means that, coming from the situation or from problems faced previously, the pupils are able to attach some meaning to every word or mathematical expression occuring in the statement. This also means that they are able to decide whether a proposed answer to a question in the problem is consistent with the question, not necessarily whether it is correct (back to the distinction between understanding and solving a question).

• *The pupils are in position to attack the problem* with their knowledge and their habits, but the appropriate tools for solving it completely are not available to them: the situation is open for them.

Some of these tools are part of the teaching project of the teacher. The pupils should be able to use them implicitly, in terms of action or expression of conviction.

• *The problem can be formulated in at least two different frameworks.* This condition is for the teacher an invaluable control lever in order to provide the pupils with the means to progress under their own responsibility (at least in part), without being bewildered by what is new or unusual. Implementing interplays between different frameworks is indeed a way to ask milestone-questions between what is known and what is expected, without being totally dependent on the teacher (which would destroy the interest of putting the pupils in a research situation). The pupils are thus given an

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opportunity to create new from available old. In this way, the new matter gets a chance to acquire meaning for the pupils and to become itself matter for work. The teacher then has to institutionalise this new matter and give an opportunity to the pupils to use it.

3.4 Interplays between different frameworks

A *framework* is a mathematical domain made up of concepts with their double status of tool and object, of relations between these concepts, of their properties and known theorems implying them, with their different forms of symbolic expressions and representations. The carving up of a mathematical field into various frameworks may be more or less refined according to the problems to be dealt with.

Consider a problem which can be formulated in at least two different frameworks, and assume that a pupil is trying to solve it while the knowledge at his disposal is insufficient. Then an interplay between different frameworks is caused by relating various formulations of the problem, chosen by the teacher for a teaching purpose. Searching for the solution to the problem leads to using some tools that belong to each of them, and to relating them. The interaction provides new questions, conjectures, solving strategies by appealing to tools or techniques whose relevance was not predictable under the initial formulation. This work leads to an evolution of conceptions, and possibly to the creation of new objects.

Requirements for interplay between different frameworks

In order for the interplay between frameworks to fulfill its expected function, we state conditions that we consider as necessary, but not sufficient:

- the pupils have some knowledge in each of the frameworks appealed to;
- their knowledge is partial in view of the needs of the problem;
- they are in a position to relate different formulations of the problem, namely those which live in different frameworks. They can establish some correspondence between the tools used, but such correspondences are imperfect. Hence: cognitive unbalance.

However, these correspondences suggest questions and conjectures whose study creates a pressure for their improvement. Progress in knowledge relies on improvement in developing correspondences and setting up relations.

3.5 A first problem

Two positive numbers a and b are fixed. Find a rectangle of perimeter $2a$ cm and area b cm^2.

This problem depends on 2 parameters: the numbers a and b.

On the other hand, the problem can be formulated in numerical terms: find 2 numbers with prescribed sum and product. We explain in which circumstances it is interesting or even unavoidable to introduce a geometrical formulation. Our objective is that pupils have some way to progress,

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namely by implementing interplays between the algebraic, numerical and geometric frameworks.

The problem above concerns the length and area of geometrical objects. When numerical values are chosen for a and b, the problem can be understood at the end of primary school (10-11 years). But it can be just as interesting to pupils at *collège* (11-15 years) and at *lycée* (16-17 years) from whom the procedures to be expected and the understanding aimed at are clearly different.

In fact, it is one of a whole set of problems. The teacher can choose values for a and b, or can decide to keep them as free parameters. He is able to fix his strategy and manage the situation as a function of his didactic intentions and of the understanding he has of each of the pupils with respect to their mathematics and their social behaviour in class.

3.6 Important questions for the teacher, preliminary to a specific choice of the problems

• What is the purpose of the situation for the pupils: new knowledge, a new method, implementing something that has already been learned, bringing together in a more complex situation things that have been learned separately...?

• What procedures will the teacher want them to use? What attitudes does he want them to adopt: initiative, checking, looking for consisten- cy ...? What ways of doing will he provide the pupils with?

• What means does he have for recording the pupil's actions that are effective, for providing an explanation for the gap between what he expects and what they do, for evaluating their understanding in terms of mathematical objects and in terms of available mathematical tools?

Depending on his teaching objectives and his expectations of what the pupils can do, he will choose a certain field of problems. In that field, he will state the problems he wants to be studied by each pupil and he will decide whether the pupils should work individually, in groups, or in a sender-receiver communication mode. He will foresee ways in which each pupil may approach the work: exchange of ideas, confrontation, diffusion, collective working. He will perhaps afterwards come to modify his predictions, or complement them, and review them before being in a position to institutionalise what was for him the main purpose of the teaching situation.

3.7 Didactic intentions: interest of the geometrical formulation

• For pupils at first years of *collège* (11-13 years old), one of the objectives is to deepen their understanding of numbers. For this purpose, we choose to test their understanding of fractions and decimals as tools for approximating as closely as they wish measures of a type that is as yet unknown to them: namely irrational measures.

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We do not want to suppose the existence of irrational numbers *a priori*. In order to begin dealing with the question of approximating irrational numbers by rationals, we need to introduce a non-numerical context where the irrational measures have a meaning for the pupils since, from a numerical point of view, these measures exist only for the teacher.

Geometric formulation in terms of rectangles will bring conviction about the existence of lengths, whose measure one cannot express but only give approximate values, closer and closer to what is looked for. The geometric formulation allows pupils to express convictions and to involve a certain piece of geometric knowledge that is available, which will eventually favour the implicit use of an 'intermediate value' property. The task will then be to translate numerically this geometric existence. The case described is unusual, since it concerns not just one number, but a whole list, and whether this stops or not is unclear. The teacher and the pupils will agree to accept that a sequence of approximate values, closer and closer to what is looked for, is a form of answer. Thus one can be more or less precise according to one's patience. They will work out an algorithm such that, at each new step, the precision is improved.

• For pupils at the beginning of *lycée* (16-17 years old), in addition to the above, the objectives include starting to deal with solving quadratic equations, and with problems depending on parameters.

• For all pupils, to implement interplays between different frameworks in which they have partial knowledge. It is possible to show the scientific fruitfulness of such a work, for instance by extending the correspondences between frameworks.

Here, the frameworks involved are: the numerical, the algebraic (including numerical functions and graphical representations), and the geometric.

We choose to work with geometric magnitudes: area and perimeter of rectangles. This presupposes that the pupils are familiar with this context.

Since we want to approach irrational numbers through using fractions and decimals, the unknown measures are described indirectly. They are present in the relations between the known measures. These translate into equations, and it is expected that the pupils have already handled expressions involving letters and numbers.

At the end of *collège* or beginning of *lycée* (14-15 years old pupils), the algebraic model for the geometric problem is an approach to quadratic equations.

The heart of the problem concerns what happens as we vary the measures. For this we need a framework which:

– allows us to represent these variations, to collect and interpret them,

– suggests choices for calculations,

– provides for checking results.

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The graphical framework does this, and the pupils need to be familiar with using graphs.

At *lycée*, the problem lets us touch on important concepts of analysis, in a particular context: numerical sequences, unlimited decimal expansions, convergence of a sequence. This provides a first step towards real non-rational numbers.

3.8 Mathematical and didactical a priori analysis of the problem

Consider first some reasons for the choice of the numerical values of a and b.

The problem has an immediate translation into algebraic terms: to solve a system of two equations with two unknowns[2].

$$\begin{cases} x + y = a \\ x\,y = b \end{cases}$$

For a and b, we choose integer values. Depending on the chosen values, three cases can arise:

- a solution exists and the dimensions of the rectangle are integers;
- a solution exists and the dimensions of the rectangle are non-integers (irrationals);
- there is no solution.

The problem is immediately solved formally by those pupils who master the theory of quadratic equations, particularly the relation between coefficients and roots. It is a matter of solving the equation $x^2 - ax + b = 0$ which is a routine exercise (but negative solutions would not fit the geometry).

Interesting questions arise in the numerical framework. We can explore the solution possibilities for different choices of a and b. For example, if a and b are rational, we know that solutions can be either rational or irrational.

– Is it possible to have solutions where one is rational and the other irrational?

– If a and b are rationals with moderate denominators (written in their reduced form), can the solutions, written in their reduced form, have very large denominators, in which case we are not very far advanced towards knowing that the solutions are rational?

But, for those pupils who do not know about the relation between the coefficients of a quadratic equation and its roots, the situation offers an opportunity for a rich investigation relating to several aspects, provides for different levels of enquiry and generates new questions in different frameworks.

[2] From a didactic point of view, we have presented here an algebraic reduction of the problem that was formulated in terms of rectangles. This comes from the fact that the conditions: "fixed perimeter k" or "fixed semi-perimeter $k/2$" define the same rectangles. It could be envisaged that initial work on the problem, and a first result of this work, for pupils at the beginning of *collège*, could be done by substituting a question with another one which is technically simpler but which does not involve the generality of the problem.

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From a mathematical point of view, it is a very simple situation where the approximation of an irrational by rationals is tackled face to face.

A method for finding the solution when it exists is:

• Choose two numbers x and y such that $2x + 2y = 2a$ or, what comes to the same thing, $x + y = a$.

• Evaluate the product $x\,y$ and compare it with b.

If $x\,y = b$, the problem has been solved.

If not, choose another better pair (x, y) and then even better ones where $x+y = a$ and the products $x\,y$ become closer to b. Thus, either we actually reach b, or we get closer to b and the difference from b gradually diminishes and becomes closer and closer to 0, or the values seem to plateau and b is inaccessible.

The frameworks

– *geometrical*: rectangles, sides, dimensions (i.e length of the sides), perimeter and area are mobilised objects present directly (perimeter, area) or indirectly (dimension) in the problem statement.

– *algebraic*: equations, unknowns, solutions of an equation, systems of equations, are tools for formulating the problem in a different way.

– *functional*: the variation of two variables linked by a relation (here an affine relation of the form $y = ax + b$) and the method of determining two variables governed by two corresponding relations, are intermediate studies for dealing with the problem in its algebraic formulation.

– *graphical*: takes elements from the geometrical framework and also from the numerical one. It provides a geometrical model of the range of problems posed, through a set of points of the Euclidean plane referred to an orthonormal frame. Certain tools of geometry like alignments of points, symmetry, intersection of curves are adapted for use in the situation. The coordinates of points are numbers, and the graphical interpretation for the recorded variations as well as the graphical representation of an affine function are aids for the intermediate study.

– *numerical*: integers, certain fractions, decimals, together with the notion of order and their operations are basic tools in all stages of the investigation.

The tools

(i) *Tools available to the pupils.* Knowledge in different frameworks:

- *geometry*: a certain understanding of rectangles and relations between dimensions, perimeter and area.

- *numbers*: a good understanding of integers, a certain technical understanding of decimals and some fractions, and also "between any two decimals, there exist other decimals" together with a way of designating such numbers. Of course, concerning irrationals, nothing is supposed known.

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- *algebra*: ability to designate by letters numerical unknown values, corresponding to contexts which make sense for the pupils concerned; ability too in the writing of relations conjugating letters and numbers.

(ii) *Practice in the graphical framework.* The representation of number pairs (x, y) linked by a relation.

Here the relation is $x + y = a$ and possibly those pairs (x, y) such that $x\,y = b$, these representations being separate or on the same graph.

(iii) *Tools to be created.*

Consider the following proposition: *Of all possible rectangles with a given fixed perimeter, the one with the greatest area is a square.*

This is a theorem to be proved and institutionalised in order to give it the status of object in the geometric framework. It is a piece of knowledge which can be reused in other problems. For those pupils who know when and how to use this theorem, it will be part of their knowledge as a tool and as an object.

(iv) Another tool is implicit in the functional framework in a geometric context: the function which relates the two dimensions of the rectangle with its area has the following property:

Among the rectangles of a given fixed perimeter, if one of the rectangles has an area A less than the required area, and another A′ has an area greater than the required area, then there exists a rectangle whose area is equal to the required area.

(v) A method that can be set up to determine the area: look for rectangles whose area becomes more and more close to b. To do this, choose pairs (x, y) satisfying $x + y = a$ and calculate the product $x\,y = b$. Among these pairs, look for two of them, whose products $x\,y$ enclose b. If possible select new pairs (x, y) such that the products $x\,y$ will provide a more restricted interval.

– A problem with the method: *what mathematical means have pupils at their disposal in order to determine an algorithm for selection of* (x, y)?

– A mathematical question: *is it always possible to choose* x, y *so as to get* $x\,y = b$?

Didactic variables of the problem

The values of a and b. Are they integers, rationals? Are they small or large?

In other words, properties and size of the numbers are concerned. In fact, depending on the initial choice of a and b the problem may have a solution or not. When it does have a solution, this may be expressed by integers, rationals or irrationals. The problem changes fundamentally for the mathematics, and for the pupils.

Several didactic choices may be proposed:

1) a and b are integers, solutions exist and are integers. For example, $a = 15, b = 36$.

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2) a and b are integers, solutions exist but are not integers. For example, $a = 41$ and $b = 402$.

3) a and b are integers, there is no solution. For example, $a = 39$, $b = 402$.

Many other numerical choices are possible for each category. They may depend on whether the pupils have calculators (simple, programmable, or graphical ones) or computers (with spreadsheets for example) at their disposal.

Management of the class

The teacher can choose to let the pupils work in groups of 3 or 4. Each group would be given one value for the pair (a, b). The teacher chooses different values and distributes these among the groups and only uses the same pair twice. The intention is to allow the pupils to compare what they do as to methods (using the same or different values of a and b) and to compare the different numerical values that turn up for two groups working with the same values of a and b. Remark: pupils may be able to handle non-integer numbers, but choosing values for a and b which are not integers complicates the technical task while adding nothing to the conceptual level of the problem. This remark is worth making when we look for a proof of the non-existence of a solution.

Procedures we may expect to be used and which take the problem forward

We are not concerned here with errors, checks or blockages which arise from random and unorganised attempts that occur when first working on the problem, but only with procedures that are thoughtful and actually observed in the organisation phase of the investigation and in sorting out mixed-up data.

First numerical choice for a and b. Starting from

$$\begin{cases} x + y = 15 \\ x\,y = 36 \end{cases}$$

look for two numbers whose sum is 15 and work out their product. Select the two making 36.

The method is equivalent if product and sum are reversed.

A small number of tests carried out on integers quickly leads to the result.

Second numerical choice for a and b. Now consider

$$\begin{cases} x + y = 41 \\ x\,y = 402. \end{cases}$$

To use the same procedure as before will take a great deal of time. There are a great many integers whose sum is 41. We would expect the pupils to organise their results into a 4 column table: with headings

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$$x \qquad y \qquad x+y=41 \qquad xy$$

and to look through for 402 in the fourth column.

But it is not there! However, we have

$$25+16=41 \qquad 25\times 16=400$$

and

$$24+17=41 \qquad 24\times 17=408$$

Now, the number 402 lies between the two, so we would expect a geometric interpretation of this numerical fact and a conviction or a question to be expressed: *there must be a rectangle between* (25,16) *and* (24,17) or: *is there a rectangle between* (25,16) *and* (24,17)*; how can we find it?*

3.9 Elements of an a posteriori analysis

In fact, a first response heard from pupils, both at the beginning of *collège* (11-12 years old) and at the beginning of *lycée* (15-16 years old), after looking through the table of results for the dimensions, perimeter and corresponding area is: 402 *does not appear in the table, there is no solution.* In reply to the question put by an observer: *do you know any numbers other than integers?* an answer is: *yes, there are decimals, but there are too many of them. You wouldn't want me to try all of them, would you?* Now, these pupils do have a certain technical competence: they know that between two decimal numbers there are many others, and they know how to work with these numbers. But they find themselves helpless when faced with a multitude of possibilities, to the point of rejecting all decimals and restricting themselves to working with integer values which they can deal with exhaustively.

To put it another way, the question for these pupils is: *how can we deal with an infinite number of possibilities?* What are the conceptual tools and techniques which they need to deal with this situation?

In order to explain the position of those pupils, we can propose the following interpretation. The introduction of decimals to the pupils as "disguised integers" (writing the lengths in another way by changing the unit, or just rules to calculate) can lead them to acquire easily and quickly a reasonable technical competence. We would expect them to apply the techniques, suitably adapted, that they have learnt from using integers, according to their disguised integers' conception of decimals : here examining all possible numerical cases in a given interval. If the list is infinite, as is the case here, the method is simply impracticable. Lacking another conception of decimals, which would allow them to search by using another procedure, they are led to reject the decimal numbers themselves and to restrict the field of investigation to cases involving integers, which leads them to be able to state with conviction that the problem has no solution. For many pupils, the conception "disguised integers" is an obstacle to understand the density property of the ordering of decimals, and to using these numbers

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to approach any measure of magnitude as closely as they want, and more generally to use them in approximation contexts.

Drawings and graphs help in making new choices.

It is possible to construct many rectangles having a semi-perimeter of 41 by a geometric method of 'compensation'. Reducing one of the sides by a small length h and increasing the other by the same length h is a process whereby a whole set of equi-perimeter rectangles can be constructed without the need to work out the numerical values of their dimensions beforehand. By putting these rectangles in order, and on top of each other, or by drawing the graph of the points (x, y) with sum 41, the information is compressed and lets us see trends (Fig. 4). A rectangle will be represented graphically by the coordinates (x, y) and (y, x), which are symmetrically positioned with respect to the first quadrant bisector $y = x$.

Remarks often overheard.

When pupils are working on this question, they often say: *lying down or standing up, it's the same rectangle, but on the graph, we mark two points.* The ordered pairs (x, y) such that $x + y = 41$ are represented by points on a straight line. There are rectangles which we cannot put on the graph because we don't know the measure of their sides. But they are in between rectangles that we have marked. Beside each point marked in the diagram, we write the area of the rectangle that it corresponds to. When the points with coordinates (x, y) and (y, x) get closer to each other on the graph, the areas of the corresponding rectangles increase. At the center is the square, which has the largest area.

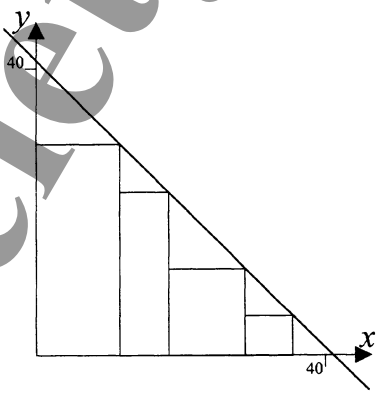


Figure 4

Reading the graph produces a conjecture: as the difference between the dimensions increases, the area reduces. The significant variable is the difference between x and y.

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The interaction between the geometric and graphical frameworks lets the pupils find and explicit the *pertinent hidden variable.*

Thus, they are now in a position to carry out their procedure effectively:

• choose a number between 24 and 25, and its partner to make up 41. This will lie between 16 and 17. Find the product of these two numbers.

• repeat the operation in order to get closer to 402.

If the numbers are well chosen, the difference from 402 will become progressively less, but will it ever be possible to reach 402? The question is left open at the 11-12 years level. However, by letting the integer k vary, and keeping the values of x and y systematically to the lower 10^{-k} approximations, we shall have a sequence of numbers which get larger. This is seen when they are written as decimals: at each step, one more digit is added to the decimal form of the number. For older pupils this process can be used to construct unlimited sequences of decimal digits, by which we can convince ourselves that they have an infinite number of non-zero decimal digits, and that they can be added, multiplied and compared. Here we have a way of introducing rational and irrational numbers.

Another procedure is possible depending upon the level of the class, their experience, and possessing a calculator.

– Draw the graphs of $x + y = 41$ and $x\,y = 402$ with the same axes.

– Study the possibly common points.

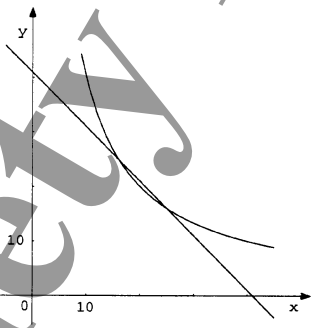


Figure 5

Third numerical choice for a and b:

$$x + y = 39 \qquad \text{and} \qquad x\,y = 402$$

• We would expect the same procedure as in the two earlier cases. Here there are no numbers in the table of results that enclose 402. All are too small.

• In order to increase the area, we can make the rectangle become "closer to the square" (this follows as a result of working on the previous problem), but even then the square has an area less than 402.

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A new question arises

Is it possible to obtain an area that will be greater than that of the square?

The teacher can propose the problem to the class: *choose a square, try to find a rectangle with the same perimeter but with an area greater than that of the square.*

The problem above is formulated in geometric terms. This is certainly a good framework for setting it, and the pupils can feel at ease with it. Indeed the context is a familiar one for the pupils and so certain ways of handling the problem will be familiar to them, notably using numbers and also transferring the problem to graphs. We would expect the use of familiar techniques when first working on the problem. These would no doubt produce assumptions on the part of the pupils. Some of the pupils will stop at these assumptions and they will lead to certainties. Others can be more inventive, and will go to examine the problem from a geometric point of view. The expected scenario is as follows:

– *numerical trials*: they will not be able to produce such a rectangle.

– *graphical representation of the pairs* (x, y): it is possible to see, as before, that as we move towards the centre (x, x) the area increases and that going beyond (x, x) the area decreases again. This should not be surprising since pairs of points which are symmetric with respect to (x, x) represent the same rectangle.

– *a geometric study of the problem*: the teacher could ask pupils the following: *draw a square and consider changing it into a rectangle with the same perimeter. Consider the effect this has geometrically on the area.*

This investigation follows from the earlier geometric study of rectangles of fixed perimeter and which enclose a fixed area.

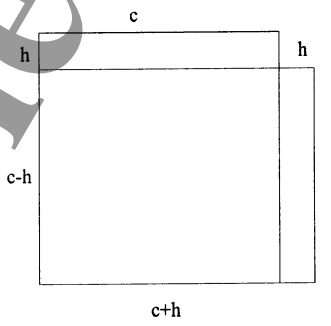


Figure 6

Now, if for the pupils an area is not simply the result of a calculation but also has meaning for them in geometry, if they are able to compare areas directly without having to carry out numerical calculations, they may convince themselves and have a geometric proof of the following fact:

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When we move away from the square, comparing the small strips (c, h) and $(c-h, h)$ that are made when changing from the square to the rectangle, we can verify that the area can only get less.

For pupils who are familiar with standard identities and with algebraic manipulation, the geometrical reasoning can be reinforced by an algebraic reasoning: a square of side c can be changed into a rectangle of dimensions $c+h$, $c-h$. Thus the area c^2 changes to c^2-h^2 and so must be smaller.

Consequence: there is no rectangle that can satisfy the given data 39 and 402. The problem is without solution.

Remark: it is tempting to assume that drawing superimposed squares and rectangles is *concrete* while drawing graphs and using algebra is *abstract* with equivalence "concrete = easy" and "abstract = difficult". We take the problem above as an example to show that we must be careful to guard against the simple opposition concrete/abstract. We prefer to talk in terms of familiarity: in mathematics, the concrete is what is familiar, what is already mathematised. This is the case for graphs: they are perceived abstract the first time, then concrete for pupils already familiar with drawing graphs and who have the competence to treat them with numerical or geometric tools. As an example of a geometric form of the problem above which is interesting but not easy to solve geometrically, consider: construct a rectangle of the same area as a given rectangle R and with the same perimeter as a given rectangle R′. A good strategy consists in transforming the questions concerning area into questions concerning lengths. The mathematical notions involved: properties of rectangular triangles, similar triangles, are taught at *collège*. But the problem is still difficult for students in teacher-training.

4. An extension of the engineering

4.1 Problem

For older pupils (at *lycée* level) we may extend the previous work, considering the system of equations:

$$\begin{cases} x+y=a \\ xy=b \end{cases}$$

where a and b are two arbitrarily fixed numbers.

In the first problem (Section 3), we have seen an example where the system has a solution in integers, one where there is no solution, and another one where, from a mathematical point of view, the solution (x, y) can be found using decimal approximations with as much accuracy as we may require. The approximations to x, as well as those to y, come together to produce an unlimited decimal expansion having an infinite number of non-zero decimal digits. This third case may be understood in different ways

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by the pupils working in a class on this problem. They have a method at their disposal, but they do not know whether it will stop eventually or go on producing decimals indefinitely. However, they have no doubt that, at each step, they improve the accuracy. And they do notice it.

The teacher allows this expansion to represent a number. He will then consider this third case as a case where the system has a solution.

For a graphical representation, two colours can be used to represent those pairs (a, b) for which a solution exists and those for which a solution does not exist.

We do not make a detailed analysis of this problem, but one can easily imagine how geometric, graphic, algebraic, numerical interactions may play.

Consequences:

• A knowledge gained from working on the first problem is revised and re-applied in the problem above:

Among all rectangles with a given fixed perimeter, the square has the largest area.

• A new knowledge arises:

The set of ordered pairs (a, b) such that $a > 0$, $b > 0$ and $b = a^2/4$ divides the set of ordered pairs into those for which the problem has a solution and those for which there is no solution. Hence:

• The system of equations

$$\begin{cases} x + y = a & a > 0 \\ x\,y = b & b > 0 \end{cases}$$

has a solution if $b \leq a^2/4$ and no solution if $b > a^2/4$.

4.2 A new algebraic problem

Is it possible to extend the criteria by which the system of equations

$$\begin{cases} x + y = a \\ x\,y = b \end{cases}$$

can have a solution, to all values of a and b positive, negative or zero?

The objectives of the problem

The geometric interpretation only has meaning for positive values of a and b. But translated into algebraic and graphical form, the problem can be extended to the whole plane.

We will not make an analysis of the problem here either. We show below the graphs which play a key role in the study of the extension of the conditions for the existence of a solution. We let the reader imagine the graphic-algebraic interactions.

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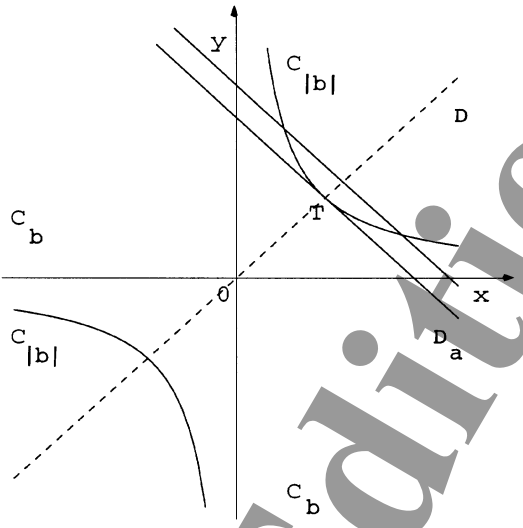


Figure 7

D_a is tangent to C_b for a particular $b > 0$, and cuts C_b for all $b < 0$.

If $b > 0$, the curve C_b is symmetric to the first quadrant bisector D with equation $x = y$. Whatever the value of a ($a > 0$ or $a \leq 0$), the straight line with equation $x + y = a$ has a slope -1 and is perpendicular to D. For a fixed choice of b, depending on the value of a, the line will cut the curve at two points symmetric about D, or will be a tangent to the curve at a point T on D with coordinates $(a/2, a/2)$, or it may not cut the curve at all. The tangent position corresponds to the case $b = a^2/4$.

If $b < 0$, the curve C_b is symmetric to $C_{|b|}$ with respect to the y-axis. Thus, if $b < 0$, any one of the straight lines D_a cuts any curve C_b in two points, one on each branch.

Drawing the graphs allows us to see that for all values of a and b the system has a solution provided $b \leq a^2/4$, and has no solution when $b > a^2/4$.

The consequence to be institutionalised is thus an algebraic method for solving quadratic equations.

In summary:

This engineering envisages involving several interplays between different frameworks and leads to an enrichment of pupils' understanding in each of these frameworks.

The geometric framework leads to a theorem: *Among all rectangles with a given fixed perimeter, the one with the largest area is the square.*

In the numerical framework: *there are non-rational numbers, representing lengths or not, which can be designated by unlimited decimal expansions.*

In the graphical framework, we see the use of a double status of graphs: on the one hand symbolising situations external to the framework, on the

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other hand, as parts of the plane, they are provided with the geometric structure induced by that of the plane. They can be a source of conjectures, a guide towards proof or a tool for checking.

In the algebraic framework, we have an algebraic method for solving quadratic equations.

Conclusion

The work presented here is an example of didactic engineering that puts on stage the *tool-object dialectic*. It also borrows important elements from Brousseau's theory of situations, like the notion of "devolution" or "didactic contract". The engineering is built starting from problems which provide *meaning* for the implied mathematical concepts. These problems can be tackled in different ways. In order to study them, pupils need to call on *different frameworks*: geometry, graphs, numbers, algebra. The *interplays* between them help the development of the synergy between the mathematical notions involved in the problem and the understanding of the pupils. The geometric framework allows an interplay among several dimensions: visual perception, inmost conviction, reasoning, symbolic representation. The confrontation between the various pieces of information obtained is a source of progress. An important place is given to the process of contextualisation, changing the context, reformulating the problem, linking questions coming from different statements of the problem, decontextualising, and also to personalisation, diffusion of procedures or personal knowings and to depersonalisation. In other words, the teacher has to act so as to ensure the change of the status from tool to object and vice versa. His goal is to get the pupils to be able to acquire knowledge which, in context, is available to them and meaningful, in order to be used directly by the pupils, or to be re-worked, transformed so that they gain a qualitatively new knowledge.

More generally, at every level the choices made by the teacher in the 'putting on stage of knowledge' are essential for the implementation of the teaching/learning process:

• First in the choice of a problem and the value of the didactical variables. The issue is to contextualize the knowledge so that it is used and acquires a meaning for the pupils.

• Next in the activities which will be proposed and the interplays between frameworks which will be appealed to, as well as in the ways of management of the class chosen in order to induce socio-cognitive conflicts.

• Then in the content and modes of the institutionalization of the new knowledge, thus having it evolve from the status of tool for solving a problem to that of object of teaching.

• Finally in the situations which will later allow reinvesting that knowledge in various contexts : direct application or more complex reinvestment.

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6. THE EVOLUTION OF GEOMETRY EDUCATION SINCE 1900

This Chapter is an expanded and more polished form of the Round table on the topic *'The Evolution of the Geometry Curriculum'*, which opened the Catania conference. The participants were: Massimo Galuzzi, Brian Griffiths, Colette Laborde and Michael Neubrand; and they described some experiences of their respective countries – Italy, Britain, France and Germany. In 1900 these were leaders in the design of mathematical curricula, and their effects persist in many contemporary curricula (see Howson [17]), so it seems reasonable to look at their systems in some detail. Because of the genesis of the Chapter, it is made up of personal views; each author has tried to get the facts correct, but there is no attempt to get a 'Committee view' on the various opinions expressed. Consequently, the relevant author's name appears at the head of each main sub-section (but the References are all listed together at the end).

Of necessity, our account forces each of us to look at some historical matters, and these lie within the Social history of Mathematics (rather than the 'pure' History of Mathematics à la Bourbaki). Nowadays some people find Social History 'dry'; but there is no point in trying to introduce new curricula until we understand why the old ones 'went wrong'. Those curricula were responses to changed conditions, and they were often very well designed for their purpose. But as the purpose changed, they could eventually turn out to be unsuitable (or even disastrous). We must study them to avoid errors and prevent new designs from suffering the same fate. Indeed (to continue this rather aggressively historical line) since Geometry has a long and interesting history, some remarks on the role of the History of Mathematics in teaching Geometry are included in an Appendix to this Chapter. The Chapter has 3 Sections, and for various reasons, the first deals with the British experience, which has been different from the others. Our description of it must inevitably include remarks on general reasons for choice of material in any mathematics curriculum. In Section II, changes in the other countries are examined, especially by looking at the evolution of some Geometry text-books, to see how these reflect the Curricula in which they appear. Finally, Section III considers some tendencies in teaching Geometry, in the last two decades or so.

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Perspectives on the Teaching of Geometry for the 21st Century, 193-234.
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CHAPTER 6

Section I. THE BRITISH EXPERIENCE

Brian Griffiths

1. Demands for change in geometry teaching: 'Top-down' and 'Bottom-up' views

We first remind the reader as to why ICMI is worried about the future of Geometry teaching. Since ICMI is an offshoot of the International Mathematical Union (IMU), which is essentially a Union of University mathematicians, it is perhaps natural that the preliminary ICMI document [21] (reprinted in the Appendix to this book) should reflect the 'Top-down', subject-based, anxieties of the IMU; those views were largely the driving force behind the design of Curricula in 1900, and the various reports that described the situation in different countries around 1912. But changes in Society are forcing changes in the curriculum, to ensure that any practical steps to affect the teaching of Geometry must also reflect the 'Bottom-up' anxieties of the mass education systems now provided by the tax-payers of most countries (in political climates often hostile to Euclidean ideals). If we wish to generate useful advice for those systems, then it is no good simply to advocate the teaching of whatever enthuses us personally. Instead, to exercise the discipline of Mathematics Education, we must try first to be clear about these often conflicting demands. They arise from both Mathematics and Society, so we first discuss these demands in greater detail.

2. The 'Top-down' Pressure.

For brevity, I will call the Mathematicians of the IMU *Theorem-proving Mathematicians* (TPM's), since they see their primary role as creators of new mathematics (a process of great joy to them); and they usually work in Universities, as teachers of young adults who will carry on their long and honourable tradition of transmitting old, and creating new, mathematics. Frequently they insist on the importance of 'thinking geometrically', which they do not specify but rather indicate by example. It seems to be not quite the same as fluency with Euclidean theorems, but rather a complex of three types of skill:

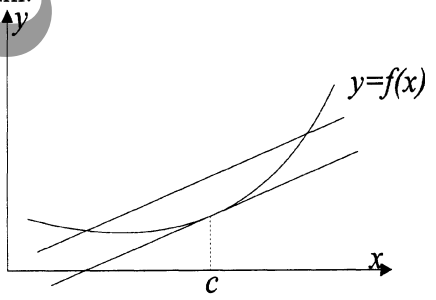


Figure 1

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• understanding theorems of Continuous mathematics in terms of pictures rather than symbol manipulation; for example to think of the Mean Value Theorem in terms of Fig. 1 (once banned by the Analysts of the 1930's (!)), or to think of the solutions of equations in terms of geometrical manifolds, and transversality;

• converting the sketches into the formal language of a 'proof' in the style of Euclid;

• acquiring a taste for the discipline of organising a theory, with Euclid as paradigm.

The TPM's have traditionally expected that the Euclidean paradigm be planted early at the Secondary level, so that it is ready for them to build on when they exposit organised theory from Algebra or Analysis (even though courses in explicit Geometry are now rare in Universities). But nowadays few University students understand the paradigm, even if they have met it; and the TPM's want corrective action to be taken, usually without realising the complexity and difficulty of what they are asking, for they are usually unaware of the History and in danger of repeating earlier errors. The problem is not new; but Mathematics and Society have both changed in more complicated ways than before.

The earliest attempt to solve this type of problem was to think in terms of the education of a tiny élite (usually males) segregated in schools such as a Gymnasium or Lycée, and to teach material to make a good foundation for University work. If a student found it too difficult and fell by the wayside, then he should not have been there in the first place, and his family would still support him. With these social constraints, people devised the curricula that were in place by 1900; but by then there was pressure for change, which we now describe.

3. The English 'Golden Age'

Since details can be found in Howson [16], and Griffiths & Howson [14], we only sketch here a brief account of the British experience (mainly English and Welsh: Scotland has its own system). Readers should remember that until the 1980's, the Curriculum was not determined centrally as in other countries: earlier religious controversies had forced a 'hands off' approach by the Government, and there was no 'official' state secondary education until 1902. From the mid-19th Century, the private English Schools, which offered restricted access to poorer children, taught Euclid for a social reason: every Gentleman should know Greek thought, which happened to contain geometry – the Theorems with (eventually) some exercises. But the subject had to be taught by a rigid exposition following Euclid's own, as demanded by the Universities of Oxford and Cambridge. Also the pupils were selected for income rather than intellect, and did not respond to the rigid rote-learning methods of teaching Euclid. It was not the best system even for the brightest, and a few teachers wanted to be more flexible in their

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teaching, but were frustrated by the rigid examination requirements. To change the system, they founded in 1871 what became the Mathematical Association.

Meanwhile new demands were arising from the users of Mathematics, who wanted the sort of geometry and other mathematics that could be used in Engineering and Physics. With the confidence of that time, they saw themselves as part of a (mythical) 'race of inventors', and demanded the mathematics suitable for men who "would earn – rather than merely spend – a thousand pounds a year". It was the use of Mathematics they wanted, not its cultural links with the Greeks. Most of them thought also in terms of teaching an élite, but a different one from before, possibly to be selected and financed by the State. The mathematical prestige of Arthur Cayley made his obstruction formidable; he was reflecting a traditional Cambridge attitude involving a Kantian philosophy of Geometry, and the place of geometry in a Christian, liberal, education (for details, see Gascoigne [10], Walters [41]). But after Cayley died, the various pressures combined with the Social change of a first expansion of State Secondary Education, and led to the 'Perry' reforms around 1905 (so-called after a leading advocate of reform, John Perry, a university Professor of Chemistry). New text-books were then written – not only in Geometry but in attempts to unify arithmetic and geometry with algebra, so that enough Trigonometry and Calculus would be available quickly, for applications in problems of school Physics (at first taught by Mathematics teachers).

It is this tradition, of trying constantly to stress the links with the real world, which has made the British mathematical curriculum development different from that elsewhere; mathematics is not taught *in vacuo*. (Traces of this philosophy can also still be found in some countries which were once part of the old British Empire.) Incidentally, some of the books of that period can still serve as models for contemporary text-book writers – e.g. the various texts of C.Godfrey and A.W. Siddons, and of C.V. Durell; and by the 1920's they had led in the teaching of Geometry to a philosophy of 'Stages' A,B,C,D and E.

Thus, for Stages A and B, the Geometry texts introduced the subject first as a branch of Physics, with practical exercises in Drawing, and avoided formal discussion of axioms, in favour of certain 'facts' of Physics. They quickly reached geometrical constructions, and then (Stage C) introduction to the proof-ritual of standard theorems on circles, similarity, Pythagoras, etc. Trigonometry was introduced early, with 'Mensuration of triangles' to calculate the heights of inaccessible objects, sometimes involving a bit of 3-dimensional work. Many of the exercises related to the outside world, in the spirit of the 1905 reformers.

(At this point, it could be helpful to agree on a standard terminology for 'exercises'. English books often called them 'Examples', following one or

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two 'Worked Examples' that gave full details of a solution; but we use 'Exercise' for something which directly develops a skill in using a construction, an algorithm, or theorem. On the other hand, a 'Problem' should mean something which requires several skills to solve, and where the choice of strategy is not obvious. In this respect, see Gardiner [9].)

All this was for a highly selected minority of pupils, with a liberal-minded scholarship system (with limited funds) that hoped to seek out deserving children from the working class. A much smaller 17-18 age group stayed at school and had then to choose only 3 or 4 subjects for the Higher examination (later called A-level). The mathematics course could then become more specialised than in the educational system of any other country. Its Geometry component included analytic geometry up to conic sections, a few special plane curves (catenary, cycloid, ...) and some 'Modern Geometry' in the sense of the 1890's, see also Davis [7]. This was Stage D, and a very small number of the most promising pupils went to Stage E, with some inversive geometry, and Projective Geometry up to the circular points; but a totally strict axiomatic treatment was avoided. Among such pupils were those who would study mathematics at University, and whose choice of 3 subjects would include Applied Mathematics (i.e. Mechanics) and Physics. Therefore they lived in an intensely mathematical climate. Most of them needed to compete for scholarships to finance their later studies. They acquired a love of Mathematics from attempting the large stock of 'scholarship' questions ('Problems' in the sense mentioned above) which involved applying known theorems with ingenuity. But these 'questions' were exclusively closed-ended: all the material was about answering rather than questioning.

Since the 1960's, this system has been greatly modified, although a few individual mathematics teachers have still retained something of its spirit. Most leading British TPM's have been fortunate enough to be fostered by it, and look back upon the system as a Golden Age, often without understanding why it has gone, or even how fortunate they were to have an excellent teacher; even in the Golden Age, there were many teachers who did not conform to the spirit of the reformers!

The "golden age" system devoted great effort to teaching geometry, especially to the 11-17 age-group. It is strange that nobody seemed to ask what proportion of 11-year olds were ready (developmentally, à la Piaget) for useful activity with Euclidean proof – the highest intellectual peak achieved by Greek adults. Even today, we have no reliable answer. We might also ask: even if a country could afford only to educate a tiny élite, would the 'Golden Age' system still be appropriate, because of changes in what scientists now need to know and to be able to do? What would be the best mix of problems to practise?

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4. Widening Secondary Education

This English curriculum survived until the 1960's, but by then there were over 20 Universities besides Oxford and Cambridge, and most had already moved away from their old courses on curves and quadrics. Now they taught linear algebra and set-theoretic topology, thus producing school-teacher graduates with little experience of 'genuine' geometry. At that time also, there was pressure to bring new topics into the curriculum, such as Probability, Networks, and Matrices; and several 'Projects' arose. A few of these were led by TPM's, but soon failed. The most successful were Teacher-led, with TPM's used only as advisers, and the most thorough-going was the School Mathematics Project (SMP) which still survives through techniques of 'rolling reform'. It began with a new series of text-books, in which there was less stress on technique with formulae; time was saved to allow earlier introduction of analytic geometry and the notion of gradient, with some transformation-geometry and matrices. The teams of teacher-writers assumed that teachers using the material would be like themselves, technically competent and interested in an open-ended approach; but the materials were widely adopted by schools with less enthusiastic teachers, who often failed to use them well. The mathematical changes were widely criticised (often ignorantly) as being of benefit only to Pure Mathematicians; and the SMP course was later altered in many ways for various reasons (through its 'rolling reform' procedures). Of these reasons, the most important was social: it had been slowly made inevitable by the 1944 Education Act which initiated moves towards Secondary Education for all, rather than for a small selected minority.

These moves, in Britain, at first widened Secondary Education by keeping the old selective system, and putting the rest of the children in poorer-quality 'secondary-modern' schools. From the 1950's onwards, the School system changed to widespread (but not universal) Comprehensive schooling without overt selection, and new problems of curriculum construction arose. In mathematics, these were met for a short time by merely taking the existing texts designed for the selective schools, and watering them down. The resulting texts were often unsatisfactory, being remote from many of the new, intended pupils – both in complexity of language and in motivation. Some enthusiasts preferred to manage with their own notes, using text-books only as something for pupils to consult. For, enthusiasts could still be found in the system, and they could work as their professional judgement guided them, because of the lack of central control. But the problems were not left to the professionals; by the mid 1970's, Populist politicians began to agitate, and an official Committee was set up to look at all mathematics teaching, excluding the University level. This was the Cockcroft Inquiry which issued a Report [6]; its recommendations were well-received by teachers, and have had some later influence. (In particular, it was the first Committee of this type to call for a 'Bottom-up' approach to

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the mathematics Curriculum, and this was supported by the London Mathematical Society, which is the national body of TPM's.) Unfortunately the recommendations were not simple-minded enough for the politicians, who soon played them down, and – now in Government – blundered on with their own recipes, causing great turmoil in the educational system.

The politicians in Britain, who took power in 1979 with Mrs Thatcher, were attracted to Monetarist theories that questioned the whole idea of free education. They were supported by certain Economists elsewhere of course, and influenced by Managerial theories of introducing numerical parameters to measure 'value'. For the first time in British history, politicians in power chose to scorn professional advice in a radical way; and the Government introduced several new laws about the running of schools, and imposed a National Curriculum. (For propaganda reasons, they always call these changes 'Reforms'.) The new curriculum applies to all pupils aged 5-16 in State, but not private, schools; and 'League tables' are compiled each year, to 'measure' the performance of each school, as to numbers of pupils who score grades in the public examinations. (Because of their small classes, Private schools usually do well in these tables, which do not describe the social circumstances of each school.) This curriculum was hurriedly imposed without pilot experiments but with frequent changes. For example, in line with the Cockcroft Inquiry, open-ended 'course-work' was at first encouraged, but this was stifled by the direct intervention of the Prime Minister (John Major – a school drop-out, with no personal experience of such assessment). Moreover, advisers were fiercely resisted if they tried to relate the new curriculum to the A-level course[1] for 17-18 year-olds; that was called the 'Flagship', beyond criticism, because it was thought to appeal to middle-class voters who wanted their children to have the qualification for social reasons. Here we see how Mathematics Education has to be aware of pressures from Society, in a way that 'Mathematics' can ignore.

A mass of Official Documents describes the general National Curriculum, but these were frequently revised after their introduction, until they were frozen by a moratorium (to last 5 years) that the Government finally agreed with the exasperated and overburdened teachers – who now must apply national tests to pupils at age 7, 11 and 14, (with the usual examinations at 16-18). The resulting mathematical part of the curriculum is hardly worthy of scholarly description, since the professionals who contributed to it were forced to rush their work without proper discussion or pilot trials. Many suitable suggestions about Practical Geometry are included, but that is the main emphasis because there is no mention of the formal notion of proof; rather – as in the rest of the Mathematics curriculum – the guidelines ask for 'mathematical reasoning', 'mathematical explanation' and 'mathematical justification', without illustrating what these terms might mean. A Section entitled "Shape and Space" requires pupils to be given "opportunities to

[1] This course leads to the 'Advanced level' certificate after two years of instruction.

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transform images and solve problems", but in practice few teachers are able to implement this vague instruction as they lack appropriate training and resources. See also Chapter 4, Sect. II,2.

Computers are regarded mainly as the focus of Information Technology, not part of Mathematics; and the Government prefers Long Division (since voters remember how it terrified their youth) to pocket calculators. The Curriculum is a poor basis for A-level work, of which the geometry section now only contains a weak form of analytic geometry. (The A-level system has always contained its own mechanism for making minor changes, and has gradually gnawed away the older geometry to make time for other mathematics; the arguments here are at least rational.) Future mathematicians are recruited from those who choose mathematics at A-level, and these will now be even further away from the requirements of the TPM's in the Universities.

In May 1997, a new Government was elected, with a very different general attitude from that of the Thacherites. But it still seems to be accepting their educational attitudes. Thus, even if the political grip can be loosened, it will be a long and difficult task to get a sensible system running again, partly because older teachers have been paid off, in favour of young teachers who are cheaper to employ. An indication of the fall in morale of teachers is expressed at the end of the following quotation from a former enthusiast (Quadling [33]) who worked with the SMP; its beginning summarises an interesting attitude about textbooks (expressed in detail in the article itself):

> The memorable textbooks are those which have been written by teachers with ambitions to reform, who believed that where they led, the curriculum ... would follow, ... They thrived on the freedom of individual teachers in the United Kingdom to teach how and (subject to loose constraints) what they wished. But the world of attainment targets, common cores and league tables is inimical to such initiatives. It seems unlikely that we shall see more textbooks to compare with the best of the 20th Century until responsibility for curriculum development is returned to its proper guardian – the experienced teacher in the classroom.

The recent British scene is not one to emulate!

5. 'Bottom-up' Pressures: the need for Practical Geometry

Suppose, however, that the British system is given the chance to change – if only to enter the next millenium. Let us consider some principles that we might expect professionals to take into account. They may seem obvious, but many TPM's are apt to forget them when planning content-oriented syllabuses. In contemporary Consumerist societies, ordinary people need to know enough geometry to do things like measuring their dwellings for quantities of materials, and marking out pieces of 'Do it Yourself' work; thinking

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ahead is needed. (From at least the Egyptians onwards, there has always been this connection between Geometry and the needs of builders.) Citizens also need to understand plans and maps, and have a spatial sense. Hence, education for all requires Practical geometry that everyone can learn, but which contains material for those who will later need more Theoretical geometry – the future scientists, etc, that the system hopes to nurture. Unfortunately we do not know which children these will be until they have been in the system some time! To avoid rushing into formality too soon, it would seem reasonable to take Practical geometry seriously, and use it as a foundation on which to build Theoretical geometry – as in the historical development of Geometry itself. We have seen with the British 'Golden Age', that even an élite of theoreticians can benefit from a grounding in Practical Geometry, but the plan was then often followed only to the extent of giving practice with ruler and protractor, before the rush to Euclidean-style proofs and exercises. Subjected to that rush, few – even TPM's – have claimed to understand what was going on until they were much older; and most of those who got by, did so largely on memory.

6. Teaching Strategies

I will now describe a response to such practical demands, in what I judge to be the spirit of British teachers before they had to cope with the muddle of the National Curriculum. True, the national curriculum does begin quite reasonably (as did Froebel teachers long ago) with much tactile work, the naming of shapes, the perception of properties and the use of children's 'natural' intuition of space to exploit these. But it seems to recognise only inductive reasoning; children need to be introduced to deduction – but in a gentle, semi-formal way at first – using 'local' logic, as with some dissection-type proofs, and geometrical constructions ("Draw a square of side 5 cm"). Nowadays this art can also use such computer packages as Cabri, and these experiences allow those who drop geometry early to have at least some useful techniques for the real world. At the same time they can form a foundation for those pupils who need more formal methods of proof.

It will be easier to give reasons for choosing such strategies, if we first recall the idea of the 'levels', which were suggested by the Van Hiele's after their own teaching experience; they form a useful language for discussion, even though they are not precisely defined (see van Hiele [17]). Briefly, to understand a new piece of Geometry, a pupil needs to have acquired certain levels (of terminology and especially experience: not just of earlier theorems as in a formal theory), and the new work cannot be understood with little or no exposure to the preceeding levels. (This seems blindingly obvious, yet some countries still have curricula that begin formally, with no contact with drawing, in attempts to be 'rigorous': why be rigorous if you have nothing exciting to be rigorous about?) Caution is needed with the highest of the van Hiele levels, since it would seem that, for the van Hiele's themselves, the

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supreme point of Geometry is its logic, whereas many other people clearly like geometry for other reasons – artistic, or through the ingenuity needed. The van Hiele's probably thought it superfluous to mention a higher level still, which is a level they themselves possessed and which is required of any 'ideal' teacher. It is the level that gives an overview of the part of Geometry that one is teaching. (Perhaps the ICMI document [21] had in mind that we should concentrate on how teachers could be led to that level.) For more details see Chapter 8, section II.

A lack of such an overview-level itself is apparent in some research into the van Hiele levels; for, the test questions often stress logic rather than the beauty of diagrams: they lack geometrical 'taste' and are boring. Indeed we need research (by mathematicians rather than psychologists) on the appeal of geometry to different types of pupil. Also discussions with my colleague Keith Hirst and a PhD student led us to argue that it is misleading to try to place a pupil at one level (globally), since they can be at different levels with different pieces of work. Moreover, the van Hiele levels seem better applied not to the pupil, but to the path needed for teaching a piece of work, as in the following exercise which the reader can try. Choose an 'attractive' theorem of geometry for a particular age-group of pupils; before you teach it, enumerate to yourself all the names and techniques that the pupils will need, in order to understand first what the theorem means, and then to form a proof. The resulting flow-diagram can be surprisingly complex, but if the teaching ignores any node or branch, what will the pupils retain?

Now, it takes time for a teacher to pay attention to the vital lower levels, so suggestions are needed for making short-cuts later on, especially if, as in many countries (excluding the USA, where there are fewer constraints), it is laid down that the curriculum must contain so much of other types of mathematics. Time itself is a serious constraint. As has been done before (e.g. in the evolution of the 'Perry' reforms) some possibilities for economy are the early use of Trigonometry (for the practical work mentioned above), as well as analytic geometry (i.e. planar equations of first and second degree, and vectors in 3D, but not "Linear Algebra" for as long as can be resisted!).

There is also one practice that needs investigation because it seems to be only superficially efficient – that of teaching Geometry alone for two or three Semesters, as in High Schools that follow US custom. This practice assumes (as do many non-teachers) that once taught, a topic never needs mention again, as if pupils had computers for memories; and of course the 'Geometry Year' ignores the need for a spiral approach through the levels, and to show the interplay between various parts of mathematics.

As for the demands of TPM's in Universities for Theoretical (Axiom-based) geometry, there is so much pressure to include other parts of mathematics (Calculus, Statistics, ...,) in the examination syllabuses, that there will rarely be enough time for pupils to have sufficient exposure to the higher

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Van Hiele levels. It would seem more useful for Universities to accept that they must themselves teach Euclidean-style 'proofs'. Their kind of 'rigour' is important, but at the Secondary level it is no more important than that of an engineer who must ensure safety or a good fit. Such an acceptance has already happened at the University of Bergen (Norway), which now gives a first-year course to introduce students to Euclidean Geometry; see the paper by Oddvar Iden in the 'pre-Proceedings' of the Catania Conference [27] pp.108-111. Similarly, the suggestions in the ICMI document about finite geometries and axiomatics seem out of the question in contemporary High Schools, even if pupils were ready (though a tiny few met this material in Britain, long ago, with unusually good teachers). At University level, such topics require Piaget-type research on mathematics that is too difficult for Psychologists: there seems to be no work that shows whether young adults are ready for various degrees of abstraction. The failure rates, or implicit watering down of standards, indicate that college students are mostly NOT ready.

7. Geometry in General Culture

We might also argue that a curriculum for the general citizen should include some *Education* – as distinct from *Training*. This was the view of Arthur Cayley and his contemporaries when contemplating their curriculum (which was for 'gentlemen' rather than 'citizens'), and also that of the Association of Teachers of Mathematics (ATM) when trying to establish a curriculum of 'Mathematics for All' in the expansion of Secondary education from the 1950's in Britain. The cultural legacy of mathematics could then convey to everyone some knowledge (if not understanding) of things like:

- the use of conics to describe the motion of planets and projectiles;
- the difference between a dogmatic assertion and an attempt to convince, preferably with examples from Law and Science; and in mathematics with a Greek proof and ones from other cultures (such as the Chinese version of Pythagoras' Theorem);
- the existence of non-Euclidean Geometries and their use as models of the Universe.

Of course, it takes time to discuss such topics meaningfully, even in a brief way. Pupils will probably not think them important if the topics are not formally examined. But this raises another topic for the millennium – how to assess work in such a way that pupils keep reflecting on it, and do not assume that they can forget it as soon as the examination is over. And this is related to attitudes in Society about the purpose of Education; the Monetarist attitudes mentioned above would be as unlikely to support these suggestions, as the Victorians who opposed the teaching of Literacy to the general population.

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8. Some Further Aspects of Teaching

So far, these remarks have concerned reasons for including material into a geometry curriculum, and the need to be clear as to whom the geometry is to be taught. The ICMI discussion document [21] ignores this, and merely assumes a captive, docile audience of pupils, who will learn whatever mathematicians lay down. But an equally important problem concerns the teaching of the material, to make it interesting. It is no good being more sophisticated than the available teaching force, so problems arise concerning the education and training of teachers. More will be said on this in Section III below.

Section II. THE EVOLUTION OF CURRICULA AS INDICATED BY DIFFERENT KINDS OF CHANGE IN GEOMETRY TEXTBOOKS

Traditionally, the teaching of Geometry has always been linked to textbooks. Although a curriculum is more than what a text conveys (because it reflects teaching style and what is spoken by the teacher), a good idea of the ways in which changes occur can be inferred from inspection of texts. These are of two kinds – those written for pupils and those for teachers. We now give examples from Italy, Germany, and France.

II.1. On the Evolution and Underlying Ideas of Geometry Textbooks in Italy

Massimo Galuzzi

1. The Cultural role attributed to Euclidean Geometry

The structure of the education system in Italy goes right back to the founding of the State, after unification was completed in 1870. The problem was not only to form a ruling class but also to build up citizens convinced of the necessity of the very existence of the Italian nation. Special care was devoted to the *Liceo Classico*, the Humanistic High School, intended for a minority of pupils and based on study of the Classics, which was to produce the new 'Italian Citizen'. What then should be the Mathematical part of the new curriculum?

The following remarks apply to that minority, since (as in most countries) geometry was, until recently, only considered as an incidental subject for the rest.

Already before 1870 different mathematical curricula were in use throughout the Italian peninsula. The elaboration of the first national curriculum for secondary schools (i.e. for pupils aged 14 - 15 to 18 -19) involved

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several outstanding university mathematicians, such as Betti, Brioschi, Cremona. The specific cultural role attributed to geometry within the curriculum for the *Liceo Classico* was singled out stating that through geometry pupils were expected to learn "reasoning, proving, deducing". Moreover, it was explicitly claimed that the teaching of geometry should follow Euclid's methodology. Thus, Euclidean geometry acquired a pre-eminent role, not only within mathematics, but even among the other disciplines which were intended to contribute to the building of the pupil's personality.

At the same time, a new type of secondary school of high scientific value was created, the so called "physical and mathematical track" of the already existing "technical schools".

2. Interpreting a liberal Syllabus

A major reform of the whole Italian school system was carried out half a century later, by the philosopher Gentile in 1923. From now on we shall discuss only the geometry taught at the 'Liceo' level. The main path for the education of pupils became the *liceo classico* (humanistic high school), and instead of the "Physical and Mathematical track" of technical schools a *liceo scientifico* (scientific high school) was introduced. Globally, the Gentile reform restricted the space and the importance of mathematics within the curricula, even in the *liceo scientifico*.

Nevertheless, geometry teaching remained fairly unchanged both in the humanistic and in the scientific tracks, with a special emphasis on synthetic Euclidean geometry, as one may see by comparing the most widespread schoolbooks of those days, e.g. Faifhofer, Sannia-D'Ovidio, De Paolis, De Franchis (prior to the Gentile reform), and Marletta, Enriques-Amaldi, Rosati-Benedetti, Severi, etc. (after the reform).

Since the end of the second world war, the traditional teaching of geometry has undergone a crisis in most countries, and even in Italy the consequences of this crisis have become apparent, although there has not been any global reform of curricula and syllabi.

Due to various factors, and specifically to the New Math movement and to the inclusion of new mathematical topics into curricula, geometry has progressively lost its privileged position within the curriculum. It has become only one among many other components which contribute to the formation of future citizens. As a consequence, since the 1960's geometry (especially synthetic geometry of 3-dimensional space) practically disappeared from Italian secondary schools.

During the decade following the New Math movement (roughly between 1970 and 1980) several innovative projects for the teaching of mathematics in secondary schools (for grades 9-13) have been tried out, and although none of these projects has become widespread, their underlying ideas are still under debate. Therefore we think it is useful to examine in some detail, in the next section, the different roles attributed to geometry within 3 of

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these projects. In the first one, which goes back to Lucio Lombardo Radice and Lina Mancini Proia, emphasis is put mainly on transformation geometry; in the second one, elaborated by Giovanni Prodi, a metric approach (close to Choquet's axiomatics) is privileged; the third one, due to Vinicio Villani and Bruno Spotorno, is more eclectic and problem oriented.

A recent attempt to update the existing curricula of all disciplines has been made through the so called *progetto Brocca* (a project, used for the last six years on an experimental basis in many schools). At this moment, it is not yet possible to foresee the evolution of this project, and the role of geometry within it.

3. Transformation Geometry

The book [26] by Lombardo Radice (a university Professor) and Mancini Proia (a schoolteacher) largely reflects the impact of the New Math movement, of which Lombardo Radice was (at least at the beginning of the 1960's) a convinced supporter. But he was also deeply linked to Italian algebraic geometry and particularly to Enriques, whose ideas about philosophy and history of science he tried to revive. So the book develops a balanced view between the modern axiomatics of Choquet and the requirements of geometrical intuition.

At the beginning of every chapter devoted to geometry, with the exception of some of the more abstract parts of the third volume, a visual intuitive idea of geometrical transformations is given by common experiences related to lights and shadows. This intuitive appeal to experience constitutes a *fil rouge* through the whole book. Geometrical objects are strictly introduced within the context of transformation geometry but the impact of the axiomatics is tempered by the presentation of a large quantity of informal material and by a progressive introduction of the axiomatics itself. This occurs by freely using 'evident' axioms (such as the uniqueness of the parallel) from the beginning until the moment (in the third volume) at which a pupil is judged mature enough to be given a more advanced axiomatic setting. There is also a precise and original attempt to give much space to the history of mathematics, but more will be said about that later.

As observed above, transformation geometry presented at the beginning largely relies on informal ideas of parallelism, perpendicularity, etc. taken from the pupil's previous experience, and it is founded upon the presence of implicit axioms left to the intuition. But these ideas progressively take a more definite shape. In the third volume, affine plane and space are presented completely axiomatically and the more informal ideas of the beginning are revisited. The book contains also a chapter devoted to projective geometry and a chapter about finite geometries.

If one considers that geometry is only a part of the book, which contains also the first elements of calculus and of algebra, with some care paid to group theory, it is quite evident that it requires an 'ideal' pupil (and

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teacher) who can devote themselves to the study of mathematics in complete freedom. But what happens if, for some reason, one has time only to cover a portion of all the geometry of the book? This is a practical problem common to any attempt at teaching geometry by a gradual shifting from an intuitive approach to an axiomatic one. If the process is interrupted it is hard to say what a pupil has gained.

4. A Metric Approach

Another university Professor, G. Prodi, uses a different strategy to introduce geometry in his book [32]. First of all he begins the book with probability theory: a clear signal of the relevance he attributes to the introduction of this subject into high school. Great care is devoted to connect probability theory to the other subjects, following the idea that not only must we avoid teaching a mathematical theory in isolation, but we should also profit by the comparisons between theories, to make their common structures stand out.

As for geometry, every axiom is justified by arguments drawn from experience, but the need is firmly argued for axioms when giving mathematical proofs. For example (pp.192-193, vol.1) the empirical description of an orthogonal symmetry is given by referring to a transparent sheet folded in the usual way, but it is observed: "Till now we have defined and described the axial symmetries: but to describe an object doesn't mean to affirm that it exists ... We need to introduce an axiom: *Given an arbitrary line R of the plane there exists one and only one axial symmetry having R as its axis.*" This quotation gives an idea of the strategy of the book: a modern approach to geometry (and to the other topics) not too far from that of Enriques, even if the geometry is very different. Real numbers are presented before any geometrical consideration is given. By their help, the axioms of distance are introduced and that is enough to explain rigorously what a circle is. Then, to characterise the straight line, a set of axioms is introduced, ending with the axiom which sets up the one-to-one correspondence between a straight line (with two marked points) and the real numbers (with 0 and 1).

Perhaps the book is hard reading for a pupil, but some kind of personal elaboration is precisely what the book demands. In the Preface, the relevance of 'doing' mathematics is stressed, and the methodology required for the teaching presented in the book is described as an 'education in thinking'.

5. A less formal approach

The book [40] by Villani (a university Professor) and Spotorno (a schoolteacher) is more informal. It is nearer to problem-solving than the other two books. But problem-solving is not interpreted as a series of problems somehow artificially constructed to arrive at a desired purpose. The idea

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is to reproduce, sometimes with the help of the history of mathematics, the context of a mathematical discovery, leaving the pupil to gradually arrive at the solution. There is not a sharp distinction between algebra and geometry, which are freely used to get reciprocal help. Also, geometrical notions are developed – sometimes informally, sometimes in a more cogent way – from the background of notions of geometrical drawing that a pupil is supposed to possess before entering into a high school. Local chains of deductions are used to describe particularly relevant geometrical facts. For example, to explain the idea of incommensurability, some propositions are stated (vol. 1 pp.123-127) as preliminary (intuitive) assumptions that can be made. These propositions can all be proved by the help of Euclidean axioms, but they all possess the common quality of a high degree of evidence (for example, "from an external point of a circumference we can draw two tangents, and the segments of tangency are equal"). Now a rigorous deduction of the property can be derived by the propositions assumed. This strategy has the advantage of avoiding the appeal to memorising practice, as the pupil has on the spot all the material that can be used to furnish a proof. Evidently, too great a use of this technique can create redundancy and undermine the idea of geometry as a global theory. But the book has a moderate use of such local chains of deductions which can provoke the need of a theory as a personal demand rather than producing an effect of confusion. A negative feature of the book is that it is really very demanding for the teacher, a difficulty which perhaps could be overcome by the help of a teacher's guide book. But the use of guide books hasn't a strong tradition in Italian teaching.

6. Books as Collections of Exercises

The approaches of these three books show that there were, at the end of the 1970's, large possibilities of teaching in many different ways. These possibilities still continue, but the pressures on teachers make it unusual for such books to be used as other than trend-setters by more commercially minded writers. At present there is a great production of big textbooks, mainly used as collections of 'exercises', having thousands of these. As so often happens, the quality of these books tends to deteriorate, and none has taken hold as a standard. The need for useful guide-lines for teachers is quite evident: are books a suitable instrument when compared with other techniques of in-service training?

II.2 Tendencies in the Changes on a German Textbook Page

Michael Neubrand

Although a detailed picture of the German school system is complicated by differences between the various Länder (separately administered regions),

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there are essentially 3 types of secondary school – the Gymnasium, Realschule and Hauptschule, for academic, technical and 'average' pupils, respectively. Traditionally, geometry in the Hauptschule concerned elementary drawing and calculations, while that in the Realschule stressed exact technical drawing and applications. Our basic concern here is with the Gymnasium, since it is largely there where geometry in the classical 'proving' sense was pursued.

Interestingly enough, in the same year 1905 when a major change in British curricula took place (cf. Sect.I, §3) the foundation of today's mathematics teaching in Germany was also laid, in the form of the so-called *Meraner Reform.* But whereas in England the changes resulted in a style of teaching mathematics that was more oriented to the real world, in Germany – starting from essentially the same body of arguments: applicability and appropriate use of calculus! – the development of teaching mathematics was more towards a closed style. The main characteristic was that the teaching was constructed over a long period not following the logical structure of mathematical theories, and also in general not following a genetic approach. Instead, it devised types or classes of tasks to be worked through by the pupils ("Aufgaben-Didaktik" in the terminology of Lenné [25]). This was the starting point for mathematics teaching after World War II.

What follows refers to pupils aged 11-19. Very roughly speaking, the development in Germany was not so different from that in other Western countries (see e.g. C.Laborde's account in the next sub-section, and Neubrand's contribution in Sect. II of Chapter 7, below). More precisely, we can distinguish three main phases of geometry teaching in Germany:

1945 - 1960/70. Traditional teaching of Euclidean geometry. The tasks included the concepts of congruence and similarity, and emphasised various constructions with ruler and compass; geometric calculations of length, area and volume were done, finally culminating in elementary trigonometry, etc. But already in that time, transformation geometry started slowly to influence geometry teaching. This was originated by raising questions from both mathematics and psychology. From mathematics, the ideas of Klein's program had become known, and Bachmann's reflection geometry showed a path to an algebraization of geometry, while from psychology Piaget's results spread out, to indicate that a basis for the development of thinking consisted of actions and the properties of composition, associativity and reversibility – all directly involved in the concept of a geometric transformation.

In the 1970's. The New Math movement, essentially an international phenomenon rather than a national development in single countries, pointed in a more compact way to the already mentioned tendencies of algebraization of mathematics, and the dynamics in developmental psychology. So, it appeared that the natural way to deal with geometry was to give

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a stronger emphasis to transformation geometry. From the psychological viewpoint a call to its intuitive nature seemed to be appropriate, while from mathematics it was oriented algebraically, since the transformations were seen as elements of the respective transformation groups; thus, their generators were studied, and transformations were used as tools in proofs, etc. (We shall come back to that question with a more recent perspective at the end of this Section.) But there was less emphasis on geometrical drawings and applications. As a reaction to the very deep changes in the curricula which attracted the 'didactical phantasy' of the teachers more than the apparently complicated geometry, there was even less emphasis on geometry overall, so that some mathematics educators in that period already believed that they were witnessing the 'death of geometry' in schools.

From the 1980's. Nevertheless, "Geometry survived, but changed" (R. Straesser, oral communication, 1995). It is not easy to characterize the present tendencies of geometry teaching in Germany. The keyword to me is 'multi-perspectivity' (cf. Graumann *et al.* [11], Neubrand [29]) which means that we find now a growing awareness of various influences on geometry teaching – mainly an acknowledgment of the fact, that it is not enough to rely on a single view, neither from the standpoint of mathematics nor from the standpoint of the theories of learning. (I will give more detail later in this chapter). So, geometry teaching today shows a complex picture: as a field for genuine mathematical experience we find orientation toward applications, as well as the more classical view of geometry; geometry with the appeal of a kind of science as well as beautiful mathematics. But the emphasis is always on the mutual relatedness of these aspects, not on a decision for the one or the other aspect.

In such a complex situation how can we best illustrate changes in geometry teaching? Limitations of space suggest that perhaps a spotlight will reveal more of the flavour of the changes than a systematic account (for that cf. Graumann *et al.*, [11]). So, since textbooks are always sensible indicators to changes in teaching, can we find the trend to multi-perspectivity reflected in the textbooks? We shall compare two pages of the same textbook "Lambacher-Schweizer: *Geometrie I*", one of the most often-used in Germany. The first edition appeared in 1969, in the period of the New Math movement, and in the course of time was further and further developed up to the 1983 edition, which is still in use. We reproduce, by kind permission of Klett Verlag, corresponding pages from the two editions, which contain the classical topic of tangents to the circle. (See pp. 212-213)

At a first glance, there are no major differences: both editions deal equally with the propositions:

- tangent is perpendicular to the radius,
- tangents can be constructed by ruler and compass, by a standard construction.

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A closer look however, reveals three main tendencies that suggest the growing multi-perspectivity pretty well, when suitably interpreted:

Tendency 1. While the 1969 edition is completely confined to inner mathematical issues, the 1983 edition shows also a photo of a hammer thrower, asking for the trajectory the hammer will take when released. So, the newer edition asks for a broader concept-formation including also experiences in fields of applications.

Tendency 2. The 1969 edition contained a series of drawings illuminating the idea that a tangent can be seen as the limiting case of a sequence of secants of the circle. This idea can be found in the 1983 edition, one page later, but as an open formulated task to the pupils: "*What is the locus of the midpoints of all segments through a circle, starting from the same point P on the circle?*". So, emphasis is now given to pupils' selfguided learning and to the possibility of pupils' own investigations and explorations.

Tendency 3. In contrast to the 1969 edition which formulates the relevant facts about tangents in a very strict and formal language, the 1983 edition gives only short, easy, common-language formulations. So, the tendency is that language should not fence in the thoughts of the pupils, but open broader fields of meaning, interpretations, and inner pictures.

But, caution is needed when identifying such tendencies. Each of them can also have the opposite effect in the classroom. So, for example, the task in the second tendency can appear in the classroom not as an open-ended question but as an unconnected exercise in giving a proof; it is for the teacher to put this task in the right place of the lesson and in a fruitful learning context. Also the picture of the hammer thrower may be just a nice illustration, not a source of discussion; and if so, the further question is, whether it then triggers the right association: are pupils really able to capture the behaviour of the hammer when released? Also the question of the appropriate language in the classroom cannot be answered without looking at the general culture there – should the language be lively and full of associative meaning but potentially misleading, or should it be correct but poor in stimulating ideas, thoughts and connections?

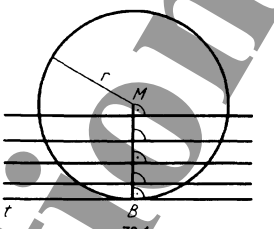
So, from an analysis of the textbooks alone, we cannot see whether teachers really support the idea of multi-perspectivity in the preparation and the performance of their lessons. However, hardly anything is known in an empirically reliable way and on a broader scale about the actual teaching behaviours in the classrooms. So finally, we can only state that textbooks open at least the possibility of broadening the perspectives towards geometry and geometry teaching. It is up to teacher training in its various appearances to foster this broader view.

Second Edition

Lambacher-Schweizer, 1969 edition.

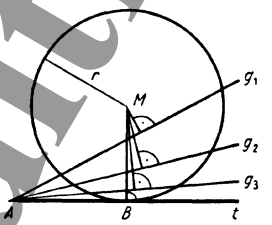
33 Kreis und Tangente

Schneide einen Kreis nach Abb. 72.1 bis 72.3 mit einer Folge von Geraden (Sekanten) $g_1, g_2, g_3, \ldots$ so, daß die herausgeschnittenen Sehnen immer kleiner werden. Welche Lage hat die Gerade, wenn die Sehne zu einem Punkt zusammengeschrumpft ist? Beobachte, wie sich dabei die Abstände der Sekanten vom Kreismittelpunkt ändern. Wie groß ist der Abstand in der „Grenzlage" geworden?



72.1

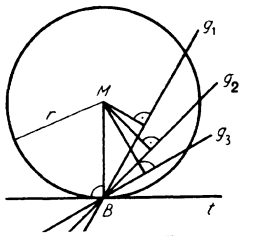
E Bewegt man eine Sekante g so, daß ihre beiden Schnittpunkte immer näher zusammenrücken (72.1 bis 72.3), so erhält man als *Grenzlage* eine **Tangente** *(Berührende) t.* Sie trifft den Kreis nur in *einem* Punkt B, dem **Berührpunkt.** Der Halbmesser $\overline{MB}$ zum Berührpunkt heißt **Berührhalbmesser.**



72.2

S 1 Eine Tangente steht senkrecht auf ihrem Berührhalbmesser.

Anleitung zum *Beweis* (72.1 bis 72.3): Der Berührhalbmesser geht aus dem Lot hervor, das man vom Mittelpunkt auf die Sekante gefällt hat.

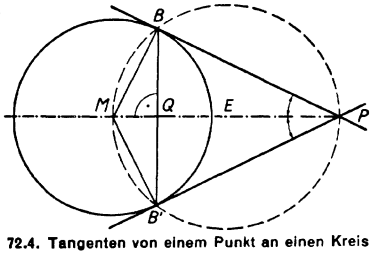


72.3. Entstehung der Tangente

S 2 Eine Gerade schneidet, berührt oder meidet einen Kreis, je nachdem ihr Abstand vom Kreismittelpunkt kleiner, gleich oder größer als der Halbmesser ist (72.1).

S 3 Die Abschnitte zweier Tangenten zwischen ihrem Schnittpunkt und den Berührpunkten sind gleich (72.4): $\overline{PB} = \overline{PB'}$.

S 4 *Die Zentrale durch den Schnittpunkt zweier Tangenten halbiert den Winkel der Tangenten und die Verbindungsstrecke der Berührpunkte und steht senkrecht auf dieser* (72.4):



72.4. Tangenten von einem Punkt an einen Kreis

$\sphericalangle BPM = \sphericalangle B'PM$, $\overline{BQ} = \overline{B'Q}$, $\overline{BB'} \perp \overline{PM}$.

Beweis zu 3 und 4: Die Zentrale PM ist Achse der Figur.

Neunte Grundaufgabe: (vgl. §§ 3, 8, 10)

Zeichne in dem Punkt B eines Kreises (M, r) die Tangente.

Konstruktion (72.4): Ziehe $\overline{MB}$ und errichte auf $\overline{MB}$ in B die Senkrechte.

Zehnte Grundaufgabe:

Ziehe von einem Punkt P außerhalb eines Kreises (M, r) die Tangenten an den Kreis.

Konstruktion (72.4): Ziehe $\overline{MP}$. Zeichne den Kreis über $\overline{MP}$ als Durchmesser. Er schneidet $\odot (M, r)$ in den Berührpunkten B und B'. Ziehe BP und $B'P$.

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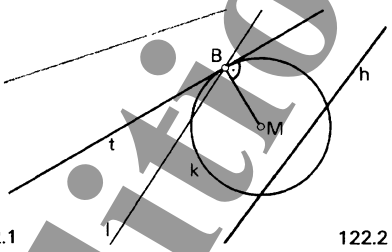
Lambacher-Schweizer, 1983 edition.

IX Der Kreis

59 Kreis und Tangente



122.1



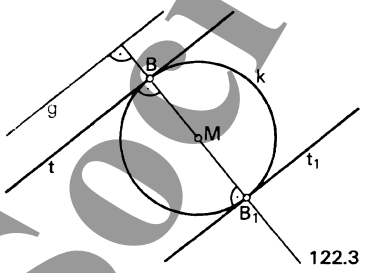
122.2

① Beim Hammerwerfen (Fig. 122.1) dreht sich der Werfer mit der an einem Draht befestigten Kugel so rasch wie möglich und läßt diese dann in einem geeigneten Augenblick los. Wann muß der Werfer loslassen, damit die Kugel in die vorgeschriebene Richtung fliegt?

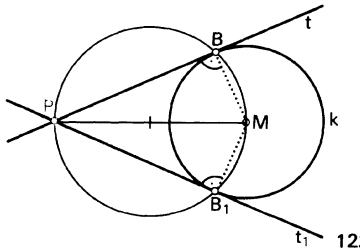
Ein Kreis und eine Gerade können 0, 1 oder 2 Punkte gemeinsam haben (Fig. 122.2), andere Möglichkeiten gibt es nicht. Bei 2 gemeinsamen Punkten nennt man die Gerade eine **Sekante** (Schneidende); gibt es nur 1 gemeinsamen Punkt B, so heißt die Gerade **Tangente** (Berührende) und B ihr Beruhrpunkt.

Satz: Eine Gerade t durch einen Punkt $B \in k(M;r)$ ist genau dann **Tangente**, wenn $t \perp MB$ ist, d.h., wenn die Gerade und der zugehörige Beruhrradius orthogonal sind (S. 32).

Diese Eigenschaft ermöglicht die **Konstruktion der Tangente**
a) parallel zu einer gegebenen Geraden g (Fig. 122.3):
Konstruiere die Orthogonale zu g durch M; sie schneidet k in B und B_1. Die Parallelen zu g durch B bzw. B_1 sind die gesuchten Tangenten.
b) durch einen gegebenen Punkt P im Kreisäußeren (Fig. 122.4):
Konstruiere über PM den sogenannten Thaleskreis; er schneidet k in B und B_1. (PB) und (PB_1) sind die gesuchten Tangenten.



122.3



122.4

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II.3 Geometry Behind the French National Curricula in the Last Decades

Colette Laborde

La géométrie est partout également ... Les objets géométriques les plus classiques (le cercle, le triangle) sont riches de propriétés merveilleuses. Les bulles de savon, les polymères, les anfractuosités naturelles sont une mine de géométries nouvelles.

(Geometry is everywhere ... The most classical geometrical objects (circle, triangle) are rich in marvellous properties. Soap bubbles, polymers, the natural fissures in rocks – are a mine of new geometries) J. P. Kahane [23].

1. Some key points

France is well known as a country that had an important debate in the 1960's about changes in the mathematics curricula. Although this was followed by a widening of Secondary education into comprehensive schools, "curriculum" was contemplated in the sprit of the academic *lyceé* – even for 10 year olds – and this account essentially concerns that spirit. The debate saw Geometry as a main point of discussion. Eight important changes in the national curricula have taken place since the beginning of the century: 1902, 1925, 1938, 1945-6, 1960, 1969, 1977, 1986 with minor changes between these dates. These changes, generally attached to a change in the objectives and aims of education, affected the geometry curriculum not only by moving some parts of the content to be taught from one level to another or changing some geometrical topics for other ones, but also by changing the philosophy of the approach to geometry. A similar phenomenon for Germany is described by Straesser [37]. Let us illustrate this through some key points.

1.1. The amazing stability of the teaching of geometry from 1902 to 1969

Geometry constituted the major part of the mathematics curriculum (about 40% as mentioned in Kuntzmann [24]). Only some minor variations occurred from one curriculum to another one such as:

- the practice of geometrical drawing (*"dessin géométrique"*) was part of the syllabi of 1902, 1925 and 1938;
- the first appearance of geometrical transformations in the curricula took place in 1925 but as "transformations of figures";
- a gentle introduction of vectors was firstly done in the syllabus of 1925 (last year of *lycée*), vectors were more used in the curriculum of 1945;
- a comparison between plane and space was expressed in the curricula of 1960 (especially about reflection).

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Inspired by a Euclidean tradition, geometry was taught with a strong emphasis on the triangle. Congruence and similarity of triangles were keystones of these curricula, following the apogee of the geometry of the triangle in mathematics. Hence Papy could claim of the geometry taught in the 1950's that Euclid and the triangle constituted its *credo*.

1.2 - The well known triumph of 'Structure' in the reform of 1969

Its promoters presented this reform as a way of ordering the chaos of old geometry which: (a) consisted of a lot of local facts, (b) required more erudition than understanding and (c) was inspired by the geometry of the triangle which died during the first world war (Davis [7]). The author of a textbook, faithfully reflecting the spirit of the reform, wrote in his preface:

L'enseignement de la Géométrie, après avoir posé un problème délicat, semble devoir sortir d'un malaise déjà long. La construction du Cours que nous avions aménagée est enfin en accord avec les nouveaux programmes élaborés par la Commission Lichnerowicz. L'introduction en Géométrie des habitudes de rigueur, de netteté des structures et d'économie de pensée, prises au contact de l'Algèbre, ainsi que de l'abandon de l'érudition localisée, sans intérêt pratique ni portée générale, constituent un réel progrès. Ce sont donc les structures, dont la connaissance est simple et formatrice, qui prédominent. [...]. Il doit s'en dégager pour le lecteur une impression d'unité profonde et de simplicité réelle. Nous espérons que l'élève en retirera le bénéfice d'une conception féconde des êtres géométriques et de leurs relations.

(Geometry teaching, after having posed a delicate problem, seems in need of moving from a rather long malaise. The construction of the Course that we have arranged is thus in accord with the new programs elaborated by the Lichnerowicz Commission. It constitutes real progress to introduce into Geometry (taken with its contact with Algebra) the habits of rigour, clarity of structures, and economy of thought, as well as to abandon localised erudition that has no practical interest or general import. Thus, those structures predominate, of which the knowledge is simple and formative. [...]. It ought to give the reader an impression of profound unity and real simplicity. We hope that the pupil will obtain the benefit of a fertile conception of geometric entities and of their relations.) Condamine, *Géométrie*, Terminales C-E, 1971.

This reform was based on the idea that it was possible to create an axiomatic geometry meant for school, and derived from the work of Klein and Artin:

Se manifestait ainsi de toutes parts le désir de disposer pour la géométrie élémentaire d'une axiomatique présentable à l'école, de nature à créer un sentiment de sécurité indispensable aux enseignants sinon aux enseignés et notamment d'intégrer l'essence même de l'axiome de Pasch sous une forme accessible et impressionante.

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(Everywhere, the wish was manifest to be able to construct, for elementary geometry, an axiomatic that would be presentable at school; and of a nature that would create an indispensable feeling of security in the teachers if not of the taught, and notably to integrate the very essence of Pasch's axiom in an accessible and striking form.) Papy.

In France the reform was guided by the underlying idea of the universal power of mathematics conceived as constructive, axiomatic and structural (the "structural approach"). It was not useful in the contemporary world to teach numerous isolated facts (especially in geometry); only the main and fundamental topics had to be taught and all the problems in various fields could be solved, as expressed by the *charte de Chambéry* of the French mathematics teachers' association:

Autour de quelques thèmes principaux, s'organise toute l'activité mathématique contemporaine et leur connaissance éclaire aussi bien les problèmes que la ménagère se pose en faisant son marché, que les théories que doit affronter le physicien nucléaire, l'ingénieur ou l'architecte.

(All contemporary mathematical activity is organised around some principal themes, and their knowledge clarifies equally the problems of the housewife at the market, as much as those which should confront the nuclear physicist, the engineer or the architect.)

The emphasis on the structural approach was substantiated by an important work of didactical transposition about an axiomatic presentation of school geometry. The debate which took place within the 'noosphère' about possible criteria for choosing between strong axioms or a minimal system is well known. As a result, all textbooks for secondary schools of that time share some common points:

- expression of a system of axioms;
- description of geometrical objects in terms of set theory: geometry was only an example of the use of mathematical structures;
- importance of the linear aspects of geometry (transformations) and of vectors;
- a clear distinction between the vectorial, affine and metric properties which was imposed by the national curricula; the affine structure was taught before the metric structure.

Descriptive geometry disappeared from the curriculum at that time.

1.3. The role and presentation of vectors

This changed dramatically in the reform of 1969 in which vectors play an essential role. But their role as well as their presentation is still changing in the various subsequent curricula. Defined by means of the notion of translation, they are now (curriculum of 1986) introduced in a naïve way (direction, orientation and length) as described by the Curriculum of 1986: "Les vecteurs sont introduits naïvement par direction, sens, longueur".

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Translations are now defined from parallelograms. And the equality of vectors is defined as a relation between a vector AB and the vector $A'B'$ (translation of AB).

As an illustration of the present decline of this notion, let us mention that the notions of *vector plane* and *affine plane* are now completely absent from the curricula. In year 1973-74, in textbooks of the last year of the *lycée* (17-18 year olds), it was possible to find a distinction between geometrical transformation and vector transformation: vector transformations were mentioned as economical tools of solution for some problems. It is no longer the case in the new textbooks of the last and present years.

1.4. The changes affecting the presentation of geometric transformations

The reform of 1969 gave a crucial place to transformations in geometry which were considered the heart of geometry. Here the emphasis was put on the structure of the set of transformations via their compositions. From the beginning until 1986 they were introduced as point transformations of the plane (or space) in the first part of secondary school. After 1986 they are presented as figure transformations at this school level.

2. Geometry and physical space: geometry and drawing

The relations between geometry, physical space and drawing have also been subject to changes, and these were more complex than may appear at a first glance. As everybody knows, the status of perceptive evidence was not clear in the curricula preceding the reform of the modern mathematics. On the one hand, some evidence could be drawn from visualisation of figures such as the famous cases of congruence of triangles which were based on physical manipulations. But on the other hand, teachers had to resist against evidence given by the diagram about three collinear points or three lines intersecting in a common point (Instructions of 1925, *classe de quatrième*, i.e. for 13-14 years old pupils). It was clear that geometry was mainly devoted to the introduction of the deductive method, conceived as the method which allows us to discriminate mathematics from other sciences. The relation between geometry and physical space, even if evoked when point, line and plane were introduced, was not part of the teaching. It appeared only with the geometric drawing and the *géométrie cotée* as expressed in the Instructions of 1938 which stated that the same teacher should be in charge of mathematics, geometric drawing and hand work (*travail manuel*).

With the reform of 1969, the role given to diagrams was reduced to that of memory of the objects of the problem, and visualisation of diagrams only seldom offered some help towards elaborating the proof. Pupils were supposed to express in an explicit way the axioms and hypotheses used in the proof and were not allowed to use some visual evidence. But geometry was not presented as independent of any reality. It was presented as offering a kind of universal model, to include both physical space and peculiar

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artifacts created to show physical instances of mathematical structures. It is striking to note that, in some schoolbooks, an effort was even made to propose to pupils physical manipulations with linkages like translators, pantographs or some hand-made devices.

As an example, let us mention the Moulton plane (IREM de Strasbourg, classe de 4*ème*, 1975, p. 143) which was a 'broken plane' made of two semi-planes on which the lines with 'positive slope' were ordinary straight lines, while lines with 'negative slope' were ordinary broken lines. It had to be constructed by the pupils themselves and was used to offer a geometrical plane (theoretical plane) not satisfying the so called axiom of dilations (A dilation is a bijective transformation of the plane transforming a straight line into a parallel straight line). This axiom requires: for every quadruplet (A, B, A', B') such that AB is parallel to $A'B'$, there exists a dilation transforming A into A' and B into B'. We must add that in this case, geometry did not model the physical device but the perception of the individual looking at the physical Moulton plane. Indeed, understanding this perception involved a lot of geometrical knowledge about orthogonal projection which was not at all explicit in the presentation of the book. In fact, the aim of the book was not to model the result of visualisation but to show the usefulness of the axiom by presenting a physical case of its invalidity.

This example illuminates very well the role given to these physical manipulations. They were *ad hoc* realities, more artificial than natural, and answered to the didactical intention to offer real situations on which mathematical interpretation could be developed. Pupils were asked to act in the reverse way to that used for creating these artifacts. It was the naïve belief that mathematics offered universal structures, that underlay this kind of practice. In the same way, some schoolbooks developed two descriptions, a description of what they called a material point, a material straight line (referring to physical space), and the description of the theoretical objects of geometry. Again, this approach was not a modelling approach, in the sense that physical space was only evoked as illustrating the theory; although the official discourse presented geometry as modelling physical space. Sometimes even theoretical formulations were applied to physical space:

> Une droite physique est un ensemble de points physiques (Instructions of 22/11/1971)
>
> Si une droite physique est munie d'une unité de longueur, elle est euclidienne sinon elle est affine (same Instructions).
>
> (A physical straight line is a set of physical points)
>
> (If a physical line is provided with a length unit, it is Euclidean, else it is affine).

The taught geometry was conceived as an example of the construction of a mathematical theory based on axioms, appropriate for school because

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elementary geometry was a closed theory from a mathematical point of view and a perfect illustration of linear phenomena. The meaning of the taught geometry was not to be found in its modelling power of spatial situations (as expressed by Bkouche several times, especially in 1983) but in the internal steps of the construction of a theory. The physical devices were intended as materialisations of mathematical structures. Geometry was not constructed as a theory allowing efficient solution of problems arising from physical situations.

The fact that geometry was not viewed as modelling spatial situations also appeared in the terminology of geometry which was taken from set theory, and sometimes everyday reference to space was avoided. For example, dealing with plane geometry, some schoolbooks called non-coincident parallel lines *droites disjointes*. This prevented the reader from evoking any visualisation attached to parallelism.

Sooner or later, numerous reactions arose in the community of mathematics teachers and educators. They stressed the importance of the figure in geometry, and the notion of configuration was proposed as a key notion of the teaching of geometry: for example, the configuration of the parallelogram refers to the spatial invariants common to all drawings of a parallelogram. They declared that the teaching of geometry should develop the ability of the pupil to identify geometrical properties from the diagrams, and recognize invariants in the various diagrams representing the same geometrical objects, so as to be able to construct diagrams by means of processes based on geometrical properties. This was clearly expressed by several papers in the *Bulletin Inter IREM de géométrie* in 1983 after some years of the new curriculum starting in 1977 for the 6*ème* (first class of secondary school, 11-12 years old). The words *dessin* and *tracé* came back into the curricula of 1977 as well as the mention of drawing instruments. As a consequence, the schoolbooks again offered more drawings. From 1969 up to now, diagrams in geometry evolve from poor drawings (few elements, all drawn elements had the same width) to more complex and sophisticated drawings. In the books of 1969, schemas were almost as numerous as geometrical diagrams: it was more important to focus on transformations as mappings than on the geometrical properties expressed by spatial properties of the drawing. The role of the diagram also changed; it became not only a memory but an object of work in the solving process of a problem, since the evolution of problems given to pupils occurred simultaneously with the evolution of the status of diagrams. Both changes were interdependent. An example in the next section illustrates the evolution of problems.

3. Examples of changes in problems from the curriculum of 1969 to the curriculum of 1977

We analysed geometry exercises given in two schoolbooks of the same grade (class of 4*ème*, 13-14 years old) by the same authors (a collective book

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written by the IREM of Strasbourg), The first of these was from 1975 (attached to the curriculum of New Math), the second one of 1979 (attached to the next curriculum). This book was not the strongest supporter of a formalist way of presenting mathematics.

The first striking difference is that no exercises in the 1975 edition gave measures, and if numbers had to be used they were denoted by letters. The problems dealt with general objects, whereas some exercises in the 1979 edition specified the length of segments, calling in an implicit or explicit way for drawing. In the older book, exercises did not mention the possibility of drawing a diagram but some drawings were proposed when the geometrical situation was more complex (as in Desargues-like configurations).

The type of problem changed dramatically from the first book to the second one. In the older one, we can distinguish the following classes of problems:

• Move from one language or type of schema to another one, Cartesian tables, Venn diagrams, arrow diagrams, writing of sets in extension, set symbolism, ...

• Combinatorial exercises on finite sets of points and lines.

• Use of invariants of transformations for determining the image of a point or showing that three points are collinear. The solution is based on deploying general theorems on invariants of transformations and on the theorem that if an element belongs to the intersection of two sets, then its image belongs to the intersection of the images of the two sets. For example:

Exercise 1. *A and B are two points whose images under a dilation are A' and B'. N is a point belonging to the straight line (AB). Determine the image N' of N through this dilation.*

Note that two cases are to be distinguished: the dilation has either one invariant point or none. A diagram could help the pupil to consider these.

Exercise 2. *Let f and g be two dilations with invariant points I and J respectively. It is assumed that $g \circ f$ has one invariant point K. Prove that I, J and K are collinear.*

• Configurations (variations on Desargues or Thales configurations) including several intersecting lines and parallel lines are given, under the form of a diagram and a verbal description. Collinearity of points or parallelism of lines are to be proved. Again a sequencing of theorems of invariance has to be constructed by the pupil to solve the problem.

• Exercises on the parallelogram requiring the use of vectors and of addition of vectors. These exercises lead to the geometrical characterisation of the parallelogram by parallel sides which were not mentioned in the book. This allowed the authors of the books to propose problems which could have seemed obvious in the past, requiring as they do an immediate application of this property.

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• Problems using the characterisation of a parallelogram by parallel sides, like:

Let ABC be a triangle with a point M on the line (BC). Find a point P of line (AB) and a point Q of line (AC) such that (B, M, P, Q) is a parallelogram. Explain the constructions you do.

Or:

Study of the image of a parallelogram by a dilation.

• Construction of images of points through point symmetry on squared paper using only a straightedge; this requires again the use of invariants in a point symmetry.

The notion of transformation and of invariants is the key notion for solving these exercises. The scope of the set of different strategies was not very important in this book. The solution lies in a subtle play on only a small number of theorems whose formulations do not refer to visual properties. The role of visualisation is almost non-existent. The order of the theorems in the sequence is more important than their content.

The more recent edition offered a larger number of exercises and a greater variety. The following types could be found:

• Computation of lengths or distances, or measures of angles.

• Construction exercises of geometrical objects that satisfy several conditions.

• Construction exercises with prescribed instruments.

These construction tasks called for the use of known properties, or in other cases for the use of images of sets of points by a transformation – as in the following problem:

Given a circle with center O and d a straight line, construct the line intersecting d in A and the circle in A' so that O is midpoint of AA'.

It is interesting to note that this latter case was not so remote from the reasoning implied by problems of the older period. But a great change was that the problem now is given in a contextualised situation. The situation is richer but more particular here, while earlier it was very poor but more general, as in the problem of the invariants of the composition of two transformations. The circle is not essential in the reasoning, it could be any set of points. The only important thing is to consider the image of d by a point symmetry and to use the theorem about the image of a straight line by a point symmetry. The point symmetry itself is not essential. It could be any dilation with O as invariant point and a condition about the ratio OA/OA'.

• Determination of the trajectory of a point depending on a moving point or a moving line.

• Computation of lengths which one cannot measure directly (*distances inacessibles*).

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• Optimisation problems or problems in which extremal conditions have to be determined, like the well-known problem of the minimal path to go to a river and take water in order to extinguish the fire of a burning tent.

The greater contextualisation of the exercises and the smaller degree of generality led to numerous possible variations and an increase in the number of exercises. Two different implicit learning hypotheses underlie the choice of exercises in each period of time. In the older one, it was assumed that details were 'noises' in the construction of meaning by the learner; the purer the exercise was, the more efficient it was to foster learning. The meaning was supposed to be attached to the structure itself and not to the situation in which the structure was hidden. This did not take into consideration the work made by generations of mathematicians for elaborating these structures, nor the fact that this type of pure situation offered very few possibilities of trials, experimentations, interactions in the solving process which are now considered as crucial factors of progress. The solution had very often to be imposed by the teacher, and the play on the theorems could become purely formal for those pupils who succeeded in achieving such play.

In the more recent past and still today, learning is assumed to be possible if contextualised. The pupil can give a meaning thanks to all details of the situation. The teaching problem is thus to allow the pupils to be able to transfer the same mathematical tool from one context to another, to be able to recognize invariants from one situation to another, to be able to change their points of view or ways of expression. The meaning is also more focused on the geometrical objects than on the structures. This is very apparent with the emphasis put on the configurations.

The diagrams, which play again an important role in the problems, offer a possibility of experimenting, although this is not really used as such in the usual teaching. This was one of the arguments given by some educators in favour of construction problems: pupils can try to evaluate their own solution on their diagrams (Marion [28]).

Another change is linked to the notion of problem. Geometrical knowledge appears as a tool in various kinds of problems involving optimisation, loci, movement, construction, computing measures,... . The 'tool' aspect of knowledge is stressed in the recent curricula.

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Section III: GENERAL TENDENCIES IN THE DEVELOPMENT OF GEOMETRY TEACHING IN THE PAST TWO DECADES

III.1 Geometry teaching needs not only Geometry, but teacher-bonding as well

Brian Griffiths

I would like to stress the need to relate our considerations to Teacher-training, especially to In-service work. Suppose we are able to recommend new curricula in Geometry: what might happen if they were adopted by school systems? It is then necessary to consider activities with students who intend to teach, and in-service work with teachers already in post. Here again, we have many problems awaiting further thought and research. To ignore them is ultimately to waste our time.

As we have noted, Teachers have to compete with the short-termism of teenage culture (and, for finance, its Government equivalent), and must catch attention by offering something that the clients find 'interesting'. Once interested, some clients may then also absorb some of the things that we think are of cultural value in Geometry, as mentioned in Section I. Now, experience from the 1960's shows that if teachers are simply told what to do by hierarchies of Inspectors, who then say "Now get on with the job", then disaster will follow. Therefore my inclination is first to change teachers, not to impose a strange curriculum upon them – guerrilla warfare, not Napoleonic full-frontal attack. Ideally, teachers would gain ideas from Conferences such as those of ICMI, and then they would band together in small groups to try things out, before building pressure to make corresponding changes in their relevant examination system. This small-group ('guerrilla') approach, can allow 'teacher-bonding' to take place, so that teachers with ideas can try them out and receive feed-back from their peers (rather than their hierarchical superiors), to discuss successes and failures. Although the SMP had some experience with this approach, the teacher contacts seem to have been too short (being essentially a spare-time activity). More success was observed in the English projects LAMP and RAMP in the 1980's, which were rare manifestations of support for the Cockcroft conclusions, by the then Education Minister (K. Joseph). Their techniques are too complicated to describe here, except to mention the crucial point that the initial groups of teachers were allowed to spend several weeks continuously in the 'bonding' process. (Taking the pay-off as a whole, that process is not as expensive as it may appear at first sight; for details see Ahmed & Williams [1], and Selinger [35].) Their approach eventually fell out of line with the growing fashion for 'Managerial' thinking of the British Government, but it is being adopted in parts of Australia and

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New Zealand. LAMP and RAMP were concerned with Mathematics as a whole (at Secondary School level) rather than just with Geometry, but we should perhaps consider whether Geometry could well be used as a central 'node', from which mathematics teaching can grow outwards, in a holistic way. Indeed, this would then be in accord with the 'Top-down' demands, mentioned in Sect. I, §1. As well as getting teachers to work together, we need awareness of some other factors relating to their teaching, as described in the next Section.

III.2 The computer as a means of finding a basis of communication with younger pupils

At the end of the Round Table session, discussion was invited from the floor. The comments of most participants were largely in the spirit of their contributions to the Conference Pre-proceedings [27]. However, it did seem necessary to point out to the participants, that we had all been speaking within the conventions of 'High Culture', yet our purpose was to consider Geometry in the 21st Century – that is to say, for present-day youth and their offspring. Most of these are ignorant of 'High Culture', because they are attracted by a different 'Youth Culture'. To be blunt, our own activity is part of the 'Academic Industry', which must now compete with what we may call the 'Youth Industry', which pumps masses of finance and effort into attracting the attention (and spare cash) of the World's Youth. The values of the Youth Industry are not our values, and its language is very different from our own. All the evidence indicates that contemporary pupils find the products of the Youth Industry preferable to our own. Therefore, if we plan the survival of Geometry, we cannot consider only possible choices of geometrical topic: we must ask (in a very profound way) whether our choice will be 'interesting'.

Some examples may help to display the subtle difficulties. In the Appendix below, Galuzzi mentions the case of the imaginary pupil who verifies some 'trivial' consequence of the axioms in a vector space, and is disappointed by the remoteness of the definitions. Galuzzi suggests telling the pupil some History of Mathematics, but (even assuming that pupils will not automatically be hostile to anything from the past) his own view of that History belongs to 'High Culture', as being part of the History of ideas. The Bourbaki approach ignores the fact that History of Mathematics can be viewed as part of Social History. I have found that even the best pupils find such exercises almost paralysing, because they do not understand the purpose of having axioms, nor are they ready to appreciate the underlying abstraction. Even within a climate of High Culture, this was always a difficult problem, being one aspect of what I have called the Problem of the Three Languages (see Griffiths [12]).

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Briefly, that problem occurs when we decide to teach any piece of Mathematics (which resides in the Literature, expressed in the 'Official' Language W of the Mathematical Community). Our target audience has a language P, which a teacher must infer by suitable probes (and 'Language' here will include gesture, diagrams, styles of argument, of telling, of listening, and attitudes of mind, the will to learn etc.). On this basis, the teacher must then develop a bridging language T, which enriches P sufficiently to allow something of the piece of Mathematics to be absorbed. In terms of a metaphor of a journey, pupils are then enabled to get off the bus (so to speak) at various stages appropriate to their ability, knowing something of the topography, without completing the whole journey. Now some teachers will assert dogmatically that W is vastly superior to P, and that the audience must accept it from the start, even if they soften the impact by going through the van Hiele Levels. True, W is 'better' for doing the Mathematics – that is how it got into the Literature – but this is a 'High Culture' point of view. Most pupils do not know that a Literature exists, and if they are told about it in an alien language, they will be left unmoved. (We have all heard of Colonial children, in the heart of Africa, being made to learn about the British Navy or French Mediaeval methods of farming.) So, the bridging language T may allow enough of the mathematics to come through, as to allow the pupils to use it and later ask for greater precision. The most important example is the way in which children acquire Arithmetic: they do not begin with the language of Peano's Axioms. Another example is the attempt to convey a language for talking about Surfaces in Griffiths [13], and the problem arises with all attempts to popularise Mathematics – a task which Humans can do (to some extent) but which appears to be inherently impossible for a computer. So this problem is highly non-trivial, even for teaching pupils who hope to acquire some 'High Culture' – as they often did when an education guaranteed a job and status.

But now we have the complication that our rival Youth Industry is infecting the pupil-language P. For example, it deliberately refrains from precision of language, because it treats a proposition as neither true nor false, but merely something with marketing potential. It stresses an Alladin's cave of jewels (i.e short-term rewards), while ignoring the work and skill (long-term reward) of those who made the jewelry in the first place. But the Youth Industry now is planting the skill of using computer technology; perhaps we can make use of that in a serious way. There now exists Geometrical software, such as Cabri, which is highly non-trivial mathematically, yet likely to be immediately attractive to modern Youth. Perhaps we can build on this, by framing desirable geometry within questions of the forms: "How can we get the computer to draw this?" or "Why do we always see that happening?". We can then share a common teaching language, that emerges from a shared interest in describing the behaviour of the friendly

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computer. But new problems concerning pupil-perceptions can arise; see Hölzl [20].

III.3 On the variety of influences on the teaching of geometry: A general list and some consequences

Michael Neubrand

Already in Sect. II.2 we pointed to the trend of multi-perspectivity as a characteristic of the present discussions about geometry teaching. This means the more explicit awareness, that geometry teaching is influenced by a variety of instances: not only by the way that geometry is seen as a part of mathematics, but also by psychological considerations and the impact of the technological change. Changes in geometry teaching in the near future may be forseen as extrapolations of trends already observable in current didactical debates. Some changes should be especially influential and will surely affect the development of geometry teaching, with respect to both the classroom teaching, and the didactical reflections. We now give a list of such influential areas, in which some recent discussions among mathematics educators took place. The list is incomplete, but indicates where we may find reasons for the recent (and maybe future) changes in the development of geometry teaching. For a more thorough discussion see Graumann *et al.*, [11]. Thus we list:

- *Changes in the development of mathematics itself.* Epistemology, philosophy, history of mathematics seem to develop towards a more complex picture of what mathematics is about. This picture includes e.g. aspects of the social embedding of mathematics, aspects of the dynamics of mathematical progress, etc. So, there is a trend in the philosophy of mathematics which is in a way fruitful for didactical adaptions and consequences (cf. Tymoczko [39] with an overview of some recent trends).
- *The greater emphasis on applications and modelling.* Applications are now seen not only as taking ready-made concepts *a posteriori*, but in the wider perspective of mathematical modelling. In that sense, applications often are seen as preconditions for the formation of suitable concepts. This wider view influences directly didactical considerations. Geometry is a very rich field in this respect, as other chapters of this book exhibit. (Cf. Niss *et al.* [31] for an overview on new trends in teaching applications).
- *The fundamental ideas discussion.* This debate about the sense of mathematics starts with a claim of the psychologist Bruner [5]. He pointed to the importance of connectivity along some guiding ideas, to provide pupils with insights into their subject. Mathematics educators entered the

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discussion (cf. e.g. Schweiger [34]) but the identification of fundamental ideas is not as easy as it may have seemed from Bruner's standpoint. Nevertheless, with this debate a complex field is open, in which mathematical, educational and psychological issues are interwoven in a characteristic manner. Fundamental ideas of geometry are seen in concepts like that of a transformation, or in actions like that of fitting together, or in general topics like homogeneity and symmetry (cf. Bender [3]).

- *Constructivist ideas of learning.* Since learning is now preferably considered as an act of a constructive creation by the learner himself, in whatever manifestation (cf. Ernest [8]), teaching changes towards forms of selfguided learning. Geometry has traditionally a great potential to foster pupils' own activities, e.g. by the richness of open problems. Complementary to selfguided activities of learning, constructivist views also emphasise communication about mathematical issues.
- *The focus on mathematics as a human activity.* As a conclusion of the preceeding tendencies, mathematics cannot fruitfully appear in classrooms as merely a collection of facts and procedures. Even in the kernel of mathematics, say when proofs are on the agenda, a wider perspective is necessary (cf. Hanna & Jahnke [15], Neubrand [30], and also later in this Section), and this has consequences which are especially clear to be seen in geometry teaching.
- *The idea that geometry is seen by pupils as an empirical theory.* A detailed study of Struve [38] reveals that a cause of friction in the classroom occurs when pupils speak about geometry as of a science with objects of reference in the empirical world, and teachers keep in mind geometry as a self-related, formal, mathematical theory. So, the status of geometrical objects in the classroom deserves more attention to understand whether pupils use appropriate arguments or procedures.
- *The impact of the computer.* Computers as a tool and as a medium now play a greater role in teaching geometry. But the status of the geometrical objects is altered, just by the presence of this other medium. That general problem affects teaching and learning sequences and the communication in the classroom, in a way that is often not easily visible (cf. Hölzl [19]). Perhaps it is in that field where the major progress, but also the major didactical problems, will arise.

So, viewed overall, the different changes show that in mathematics education and especially with respect to the problems of geometry teaching, the growing awareness of multi-perspectivity does not allow any one simple solution. We see a variety of mutually dependent influences for the future. Nevertheless, this view has some direct consequences for several classroom activities.

For further illustration in the limited space available, we again give just two spotlights on how the trend, to acknowledge the multiple influences

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on geometry teaching, actually affects thinking in mathematics education. These spotlights focus respectively on the discussions about transformation geometry in Germany, and on the meaning of multi-perspectivity with regard to the problems of proof in the geometry classroom.

The discussions about the didactical program of transformation geometry in Germany are very old (cf. Bender [2]). Major phases were:

- *Since 1900.* A debate with surprisingly broad and complex argumentations on what the advantages and shortcomes of the use of transformations in the classoom may be.
- *In the 1960's and 1970's.* A nearly ideological debate: It was asserted that only one of the possibilities – Euclidean style or transformation approach, is good, and since the latter admits algebraic precision, then transformations should be used exclusively! But critical empirical investigations revealed that pupils did not really understand the essence of the transformation proofs.

Today, apparently as a consequence of the trend towards multi-perspectivity, we have a more differentiated discussion: Some conditions must be satisfied if the use of transformations in the classroom is to make sense. Such conditions are e.g.: (i) *pedagogical* – the class is well prepared for and has a sound understanding of the concept of a geometrical transformation; (ii) *content-related* – the figure to be analysed has an inner symmetry which may reveal something interesting; (iii) *mathematical* (if mathematics is seen as a means to enhance our human knowledge and understanding) – the idea of transformation sheds light on the geometrical problem in a productive way, and fosters understanding.

So undoubtedly, we are now in a phase of really interweaving the different aspects of the transformation geometry approach in the classroom.

For brevity, let us use the term 'proving' to mean teaching and learning aspects of mathematical proof. While proving is confronted with 'mathematical complexity' when seen as a means of mathematical research, proving in the classroom is confronted with the even more complicated 'epistemological complexity' (Hanna & Jahnke [15]). To make it more concrete and to point to the consequences for classroom use, we should therefore see proving in the even wider perspective of 'dealing with theorems' (Neubrand [30]). In this view, proving is embedded in four fields of activity which must be seen as connected and which include mathematical and didactical considerations equally:

- *Introduction to the problem*, i.e.: clarifying the meaning of the problem for the pupils; referring to applications in the sense of a 'pragmatic foundation' (Hanna & Jahnke [15]); experiments; opening of suitable contexts; etc.

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• *Proving in the narrower sense*, i.e.: looking for connections; identifying suitable knots in the logical networks; looking for different proofs; convincing oneself and others; etc.

• *Practising the theorem*, i.e.: searching for equivalent formulations; rediscovering the theorem in other contexts; applying the theorem in situations other than those of the introduction; etc.

• *Using the theorem as a tool*, i.e.: using it as a starting point for further investigations; application of the theorem; etc.

Obviously, this view has very concrete didactical consequences. So in the classroom, when starting with the activity of proving, there cannot be a discrimination of practice and experience; we need room for more than the one proof, with time and opportunity to reflect upon what was done; etc. Again, this wider view is only thinkable if one bears in mind that multiperspectivity is an appropriate view of mathematics education in general, and of geometry teaching in a special and characteristic way.

Appendix. INFLUENCES OF HISTORY AND THE GENERAL SOCIAL DEVELOPMENT ON THE TEACHING OF GEOMETRY

A.1 On some relations between history of mathematics, and geometry teaching

Massimo Galuzzi

In recent years there has been (particularly in Italy, but also in other countries) a growing interest towards the use of the history of mathematics to improve mathematical teaching(1). Sometimes history of mathematics has been explicitly advocated, sometimes implicitly, as in a well-known paper by Freudenthal(2) wherein the author campaigns for an active role of the pupil in the learning process.

(1) See what F. Furinghetti says at the beginning of her paper *The ancients and the approximate calculation: some examples and suggestions for the classroom*, The Mathematical Gazette, 76 (1992), 139-142. But the problem actually is very old. In 1904, at the International Congress of mathematicians held at Heidelberg, it was suggested to introduce history of sciences into high schools (cf. F. Furinghetti, *La didattica della matematica nei Congressi internazionali (1897-1936)*, Lettera Pristem 13 (1994), 24-30). T.Heiede, in *Why teach history of mathematics?*, The Mathematical Gazette, 76 (1992), 151-157 gives a long list of bibliographical references.
Every science has its own relation to the past, and the necessity of history has to be proved not by generalities but by considering its distinctive features. Aristotle's physics surely was a great achievement, but there isn't a single statement in it which can be placed in a modern textbook, while in Euclid's Elements there are plenty of theorems which are just the same as we can find in modern textbooks. E.g. the theorem on the existence of an infinity of prime numbers.

(2) H. Freudenthal, *Geometry between the devil and the deep sea*, Educational studies in mathematics 3 (1971), 413-435.

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1. Some roles for History. Pupils need to be guided by their teacher, who must take advantage of their historical experience[3]. The use of history may be very helpful, at least to show how mathematics fits into the broader pattern of our culture. But history may also have a greater role, for example to make easier the understanding of difficult mathematical topics, by showing the increasing generality they came to acquire in the progress of time.

I could list many other 'good' uses of history, but I want also to point out some unwanted side-effects that a naïve use of history may have.

Sometimes history of mathematics comes on the scene to correct some didactic mistake, rather than in its own right. It is quite evident that this use of history may produce a bad history as well as a wrong didactic practice.

When mathematics is considered a "completely free creation of human will" which has to be taught in complete generality, even very simple definitions may appear difficult to understand.

Suppose we give a bright pupil (of the last year of senior high school, for example) a very general and abstract definition of vector space purely in terms of linear algebra. Our imaginary pupil may be able, by using the axioms, to check some trivial identity, and may arrive at an understanding of what linear dependence or independence in a vector space are. He may be astonished by the heaps of vector spaces that there are in the world, ($\mathbb{R}^n$, polynomials, continuous functions, etc.) But shouldn't we suppose that our pupil may be somehow disappointed by the definition we gave him? Is the real meaning of the definition clear to him? We can appeal to history for improving his understanding. We cannot tell the whole story of Hamilton and the quaternions, Maxwell etc. But we can point out some elements of this story: three dimensional vector spaces popularised by Gibbs and Heaviside; Peano's description (after Grassmann) of vector spaces; the need of a unifying point of view to deal with all the cases (the impact of functional analysis which led to the actual definition as a bye-product may be indicated, at least in a broad outline). At the end of the story our ideal pupil surely has understood the meaning of the concept of vector space.

But couldn't we make clear from the very beginning the aim of our definition? We might spend some time upon the fundamental example of 3-dimensional space before the general definition was given. We might emphasise how easy and natural the passage from $\mathbb{R}^3$ to $\mathbb{R}^n$ is. How great an insight we obtain by considering a polynomial geometrically as a point in a vector space (just think about the problems of best approximation...). We might make the pupil personally feel the need for a general definition.

(3) H. Freudenthal, reference in footnote (2), p.417.

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2. History and Bourbaki. It is not history which is at stake here. Or better, history is used as a tool to correct a too severe idea of mathematics(4). Another problem is that when we want to use history of mathematics as a tool to improve mathematical teaching, some attention has to be paid to adapt history to the kind of mathematics which is taught.

When we think of history for teaching purposes we are not thinking of using the results of specialised research, but of the possibility of getting help by something like the contents of a good handbook. But every handbook, even though it merely intends to describe 'facts', largely depends on some conception of mathematics the author has. One of the finest books for teaching mathematics in Italian high schools surely was the one by Lombardo Radice and Mancini Proia, I mentioned before. It is a book which has a long and careful historical note at the end of every chapter. But I think it is a good book, not because of these historical notes, but somehow independently of them. These notes, on the whole, reflect a traditional view of history, while the mathematics in the book is largely inspired by New Math. And let me explain by an example how that can create some problems. Traditional analytical geometry, merely devised as the method of co-ordinates, never acquired great estimation by Bourbaki. Far from attaching relevance to Descartes' work because of the co-ordinates (which are more or less present in Greek mathematics in Bourbaki's opinion), Bourbaki emphasises Descartes' discovery of the structural similarity between geometry and algebra, and highly praises Descartes for what he has done in the field of 'algebraic geometry'(5). Therefore the pupil may become somewhat puzzled when reading about Descartes in the note, at the end of a chapter of the book considered; for it is devoted to explaining the advantage we can gain from representing the points in a plane by ordered pairs of segments.

The pupil may then ask why we should praise Descartes for such a naïve idea. And if he hears a philosophy teacher quoting Descartes himself, who claimed that all his philosophy was based upon his mathematics, what should he think about this philosophy? Wasn't it a poor thing, after all?

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(4) Surely Bourbaki produced a very good history of mathematics in the last years, far better than any kind of history we had before. If we can look now at history of mathematics as a discipline which has reached standards of rigour which are near to those of mathematics, that is (in my opinion) mainly because of the work of Bourbaki. See also A. Weil, *History of mathematics: why and how*, Proceedings of the International Congress of Mathematicians, Helsinki 1978, pp.227-236.

(5) A. Weil was one of the first scholars to consider Descartes' method for finding tangents to algebraic curves a great achievement rather than an unfair machinery, as many historians previously described it. Same reference as footnote 4.

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The problem is that if we want to teach modern mathematics, we cannot continue to maintain traditional attitudes to such historical associations as Descartes and co-ordinates. Rather, we must, as Bourbaki indeed did, develop a history which may be set beside the mathematics we want to teach.

A.2 History of mathematics as a kind of educational laboratory

Michael Neubrand

A careful interpretation of the history of mathematics can help us to avoid certain mistakes and to get alternative ideas when designing teaching processes. In that sense, history of mathematics can be used as an educational laboratory, i.e. as a field in which educational experiences can be studied and related to the classroom teaching. Some key issues in that respect may illustrate this use of history. Through looking at historical developments one may:

- foresee mistakes of the pupils, and obstacles in the adoption of a concept (cf. Sierpinska [36]: *epistemological obstacles*);
- recognize the basic ideas and the leading problems which are at the basis of a concept (cf. Jahnke [22]: history in the classroom is a kind of "doubled hermeneutic circle", i.e. one has to interpret the interpretational work of a researcher);
- judge the importance and the value of specific topics.

In a sense, this is a reverse way of thinking about history in the classroom, compared to an only uncritical use of historical facts, events and results.

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7. CHANGES AND TRENDS IN GEOMETRY CURRICULA

It would be convenient to know the dynamics of curricular changes. What brings them about and how can changes occur? Even though it seems hard to say anything of a general nature on these questions, it is a fact that good ideas cannot always be implemented. There will always be boundary conditions determined among other things by what teachers are prepared to teach and what changes governments are prepared to pay for. To make significant changes in a curriculum the level of knowledge and the level of educational and pedagogical consciousness of teachers are decisive factors. It is not enough to have a few enthusiastic and very competent teachers. Eventually, a well qualified teacher is needed in every classroom in every school. In this respect the change of a curriculum has to start at the teacher-level. First one must train a sufficient number of teachers to master the new material. Altogether it takes years of dedicated work to change a curriculum in a subject like mathematics. In this chapter we are primarily concerned with the curriculum in geometry. It goes without saying that introducing more geometry will reduce time for other mathematics, and hence there is a need for an integrated curriculum in mathematics.

The chapter contains a general part and five invited essays. The general part is mainly devoted to didactical considerations in connection with the construction of a curriculum in geometry. We are very reluctant to make detailed suggestions about a specific curriculum in geometry. The five essays describe the current situation in Colombia, Egypt, Poland, China and Germany. Hopefully this can act as inspiration and it will become clear that there is no canonical way of teaching geometry. At the end of the chapter we shall be looking towards the 21st century.

Section I. GENERAL CONSIDERATIONS ON CURRICULA DESIGNS IN GEOMETRY

Vagn Lundsgaard Hansen

1. Perspectives for renewing the teaching of mathematics

The downgrading of the teaching of geometry at all school levels, beginning with the algebraization and formalization of mathematics in school in con-

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nection with the New Math wave in the 1960's, has led to the situation that a talent for teaching geometry might have been lost. Many younger teachers these days have had a poor training in geometry both at school and at the teacher training colleges. Rebuilding a sensible curriculum in geometry is therefore urgent. Nevertheless, we shall not argue for a revolution since teachers are not prepared to implement it.

On the other hand it is important that changes do not occur so slowly that teachers never experience a change in their own teaching.

Even though our concern here is geometry, it is clear, of course, that geometry cannot be viewed in isolation from the other mathematical disciplines when discussing curricula. We therefore want to emphasize that to renew existing curricula in mathematics is a very complex task, which among others must take the following perspectives into consideration:

- the epistemological perspective, i.e. the constantly changing picture of mathematics and mathematical activity;
- the pedagogical perspective, i.e. the existing knowledge of the process of learning and the functioning of teaching;
- the technological perspective, i.e. changes in technological facilities for doing and learning mathematics;
- the political perspective, i.e. changes in the general problems of society.

2. Objectives for the teaching of geometry

Geometrical figures are such an integral part of everyday life that geometry fruitfully can be introduced from the first grade, and even in Kindergarten. In the teaching of geometry the objectives to strive for include among other things:

- to establish knowledge of plane and space;
- to prepare pupils for applications of geometry;
- to present milestones in the development of geometry;
- to develop skills and abilities in pupils;
- to strengthen logical thinking and deductive reasoning.

It should be stressed though that the aims in teaching geometry vary from country to country, caused among others by the variations in the reading level but also on other factors not so clearly identifiable. In particular the levels of deductive reasoning vary strongly and in many countries mass education has resulted in deductive reasoning almost disappearing. In part, this is due to pupils requiring what they learn to be more 'relevant', i.e. to be more easily digestible and linked to their own experiences.

In many third-world countries the education stops somewhere at primary school level for a substantial number of children. Available statistics show

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that up to 2010 many of these countries will not be able to provide the resources necessary for an improved primary school education and especially not for a comprehensive secondary school education of the population. Nevertheless it is convenient to see the primary school level and the secondary school level together from the curriculum point of view as we shall do here. At the tertiary level, the possibilities are abundant and the traditions so different that it is hard to give detailed recommendations for a curriculum in geometry.

3. Remarks on the objectives for the teaching of geometry

Knowledge of plane and space includes the exploration and discovery of the main properties of basic geometric figures and configurations in Euclidean, topological and discrete settings using all possible tools. At an advanced level, the study of new types of curves and surfaces with the aid of the computer should be encouraged. A constructive approach in which one relies on parametric rather than algebraic equations can be recommended. Notions like symmetry, linearity, convexity, connectedness and continuity should be studied with emphasis on simplicity, depth and surprise.

Applications of geometry are plentiful. Here we shall mention only a few general types of applications, which we find particularly fruitful from a didactical point of view, namely: Optimization problems (in the Euclidean as well as in the discrete setting); problems from graph theory; geometric inequalities; geometry of numbers; interpolation problems; problems from fixed point theory; applications of linear algebra; special configurations; open problems (in very particular cases). Obviously, some of the applications mentioned can only be presented in detail at an advanced level. In the presentation of applications of geometry one should also make a proper distinction between applications within other mathematical fields and applications in subjects other than mathematics.

Milestones in the development of geometry certainly include the theorems of Thales and Pythagoras, the work of Euclid, the theory of conic sections, the idea of analytic geometry, Euler's formula for polyhedra and the discovery of non-Euclidean geometries.

Skills and abilities, which one would like to see developed in pupils, include: The ability to use different approaches for solving problems; the ability to formulate reasonable conjectures about geometric solutions and/or the skill to do rough estimates of numerical results reasonably quickly; the skill to provide models for the solutions of concrete problems; the ability to 'interpret' new configurations and new geometrical situations; the ability to manage large amounts of data. Furthermore one should strive to develop visual understanding and recognition, to improve intuition and to stimulate creative thinking. In this connection also the use of computers may prove

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quite useful. All this may sound overwhelming, and we have to admit that it is not an easy task to get pupils to acquire these skills.

Logical thinking and deductive reasoning can be related to such activities as finding 'good' strategies for the solution of problems, in looking for generalizations of results starting from particular cases, in searching for different proofs of various statements, in trying to formalize step by step parts of the theory, in looking for a uniform coherent organization of the theory, and in studying non-Euclidean geometries. It goes without saying that logical thinking and deductive reasoning should only be stressed partially and locally in connection with compulsory education.

4. The present curriculum in geometry

Until around 1970, traditional Euclidean geometry had a relatively strong position in the mathematics curriculum. It was to a large extent an object-oriented geometry with comparatively little weight on axiomatics, occupied with constructing figures and comparing magnitudes of figures. Up to 7th or 8th grade the intuitive approach to geometry usually was completely dominating. In later grades deductive geometry was introduced. In the period 1970-80, the emphasis shifted to a structure-oriented geometry putting more weight on axiomatics in affine and Euclidean spaces.

In recent years a problem of lack of coherence in the curriculum in geometry has been felt more and more. The teaching of geometry contains small bits of polygon classification, some formulas to measure various shapes, some incidence geometry, a little mentioning of transformations, a few constructions, selected loci, introduction to vectors, and finally some analytic geometry. From the point of view of the pupils, it might very well look like a kind of inconsistent 'bazaar', but we have only vague empirical evidence for this. From the point of view of mathematics educators, the stronger emphasis on self-guided learning calls for isolated and well defined problems rather than long chains of inferences. Furthermore, if we compare geometry to algebra, it needs about 10 axioms to define an ordered field whereas it takes about 30 axioms to lay down Euclidean geometry in Hilbert's *Grundlagen der Geometrie*. Hence it is no wonder that coherence is more easily lost in geometry than in other fields of mathematics.

There is no doubt that we are in bad need of a few clear guidelines to structure the curriculum in geometry. In selecting those guidelines, the question is whether one should take for granted that we want no more *early* axiomatics, that we want to remove the discontinuity between intuitive and deductive geometry and that we want knowledge to be constructed, in the sense of Piaget, by pupil's own activity. Knowing all too well that this has far reaching consequences for the teaching we shall nevertheless take this for granted here.

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5. A few main guidelines for the teaching of geometry

It seems an impossible task to suggest a core curriculum in geometry with the hope of having it implemented in all countries. As already indicated it is more fruitful in that respect to suggest a few main guidelines for the teaching of geometry throughout the school system. We can offer two such main guidelines, namely

- the study of magnitudes (estimation, calculation, measuring),
- the study of plane representations of spatial objects.

There are good reasons for the above two guidelines.

Magnitudes are part of everyday life, we use more and more measures, estimation of magnitudes are important in all natural sciences since the time of Galileo, and magnitudes lie behind the construction of numbers.

We meet plane representations of spatial objects at an early age in drawings, we live in a world of plane representations in the forms of pictures, screens etc., and plane representations are constantly used by those technicians who construct things. At a higher level the underlying method of perspective is a path to projective geometry.

6. How to conceive new curricula

In the building of a curriculum in geometry one should distinguish between the *intended* curriculum and the *implemented* curriculum. But who should conceive new curricula in the future? Mathematicians alone cannot do that: the New Math episode clearly showed that the top-down model is wrong. Teachers alone cannot do it either: they do not know enough mathematics and hence the bottom-up model is wrong. The right approach must be mathematicians and teachers working together, closely collaborating, not just consulting each other from time to time. Both groups have a lot to learn from each other. It seems, however, difficult to point out good exemplary cases, where such a model of collaboration has worked. The English SMP is an example (see Chapter 6, Section I, §4).

In the last decades, important knowledge about the processes involved in learning mathematics has been obtained within the scientific discipline *mathematics education*. This knowledge could be instrumental in bringing teachers and mathematicians together on a more reflected and conceptually equal basis for the purpose of implementing new curricula in mathematics.

7. Basic elements in a minimal core curriculum in geometry

There is not a unique minimal core curriculum in geometry that all educators in all countries would agree on. There seems however to be a number of matters and didactical principles for teaching them that are more pertinent

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than others. We list some of these matters here as a source of inspiration for constructing a detailed curriculum.

As our starting point we take the basic assertion expressed by Freudenthal that *geometry is about grasping space*, not only with eyes and hands, but also with the mind. Accordingly, the core curriculum in geometry ought essentially to help pupils in the grasping of space. If the core curriculum does that adequately, it is our hypothesis that it also prepares correctly those few pupils who eventually will decide to learn a lot more mathematics.

We shall mention below three matters which we feel should be included in a curriculum in geometry covering the primary and secondary school level.

(i) The teaching of geometry should incorporate the study of the most fundamental phenomena of metric, affine, projective and topological geometry. While metric and affine geometry may be treated in a more or less systematic setting in upper grades, this is out of reach for projective and topological geometry; what is meant, are just intuitive aspects, e.g. basic properties of the angles in some plane configurations, awareness of the visual phenomena of perspective, some combinatorial formulae like Euler's theorem on polyhedra, and the like. Those facts should be incorporated which are learnable and help in "grasping space". This includes an introduction to curves and surfaces, and some graph theory. It goes without saying that one should of course make no distinction between metric, affine and projective phenomena in the early grades.

(ii) In the early grades, teaching of geometry should be oriented towards the study of figures as well as simple motions and transformations; those which occur around us and are part of daily life. In this connection, plane representations of spatial objects are an important type of transformation.

(iii) The teaching of geometry should include links with the construction of rational numbers and an idea of real numbers in the context of geometry. It is our assertion that numbers cannot be learned through sets, equivalence relations, etc. So magnitudes (in Euclid's sense) are indispensable: Comparing and adding magnitudes (not measures), cutting magnitudes into equal parts, operating on magnitudes with natural numbers, fractioning magnitudes, measuring magnitudes, combining measures, measuring triangles, rectangles, circles, parallelepipeds, etc., up to recognizing the 'impossibility' of measuring the diagonal of a unit square. All this is fundamental geometry.

8. Selection of new matters to include in a core curriculum

Among the *insufficient criteria* for the selection of new matters to include in the core curriculum of geometry are the following:

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- it is new and important in mathematics,
- it has many applications in other sciences and in technology,
- it is teachable (not every teachable matter should be taught),
- it is beautiful.

It is more difficult to point out *necessary and sufficient criteria.* But certainly a good argument for introducing new subject-matter in the core curriculum of geometry arises if

- it contributes in a significant way to the grasping of space,
- it is learnable (not the same as teachable).

It is an interesting and complicated question how new insights in mathematics disseminate into other sciences. It is an even more complicated question how new subjects are introduced in school or how old subjects disappear. However, it is probably safe to say that in the past, new matters were incorporated into the core curriculum in mathematics first of all because they were interesting and useful to everyone, and simple enough to be within the reach of everyone.

To illustrate this point we mention as an example the system of numeration using the decimal point. It was a great mathematical discovery and bound to be popular because it is useful and simple.

As another example we take the case of functions, which are a late product of the history of analysis. The notion of a function is an extremely important mathematical concept. Nowadays functions are used by anyone because they are useful, interesting and conceptual.

If on the other hand we consider notions such as limits, continuity and vector spaces it is debatable whether they belong to the core curriculum, at least in their most rigorous forms. Surely they are brilliant mathematical inventions and indispensable tools for the mathematician. But at least one condition is lacking for introducing them in the core curriculum of mathematics without hesitation: they are too difficult and too far away from common sense and knowledge.

Elsewhere in this book arguments are given that matters like graphs, minimal surfaces, fixed points, chaos, fractals, dynamical systems can be, and to some extent should be, introduced in the curriculum to show that mathematics, and in particular geometry, is a subject very much alive.

9. The role of deductive reasoning in geometry

In the lower grades the teaching of geometry traditionally has been based on an informal, intuitive and unrigorous approach. And then, from a given grade at school, the teaching has shifted to present a formal, axiomatic,

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deductive and rigorous geometry. We think that the time has come to reconsider whether this is the right way to present geometry to future generations.

In the past the teaching of geometry in many countries has consisted in deducing a lot of theorems supposedly with the idea that pupils should learn how to recognize a proof. However, it seems not to have worked out that way.

Another approach to deductive reasoning in geometry is needed. Instead of presenting a lot of theorems to the pupil, as if they were material things to possess, it is better to present deductive reasoning as a way of thinking.

Already at Kindergarten one can point out geometrical justification for mathematical facts like those expressed in the sentence: "If I put one more cube on my tower, it will be higher than yours!"

We claim that an informal approach to geometry can be both serious and well argued. In particular, one can refer to the notion of *visual geometric structure* of D. van Hiele-Geldof [5], or to the *levels of rigour* as described by H. Freudenthal [3], who argues that there is an appropriate level of rigour for any question.

Consequently, we see no reason for an abrupt distinction between an informal and a rigorous presentation of geometry. In fact, we support the following recommendation of E. Wittmann [8]: *One of the principal tasks of mathematical didacticians in the near future is to write informal mathematics rigorously.*

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Section II. EXAMPLES OF CURRICULA DESIGNS IN GEOMETRY

In the following we shall exemplify important aspects of building a curriculum in geometry through five essays describing the current situation in Colombia, Egypt, Poland, China and Germany. These examples are of course not exhaustive; geometry curricula of several other countries would deserve attension, too. But space constraints of the present volume have forced us to limit the number of countries under consideration.

Dynamic Geometry in the Colombian School Curriculum

Carlos E. Vasco

The purpose of this paper is to consider the type of active or dynamic geometry proposed by the Colombian Ministry of Education in the mathematics curriculum for grades 1-9, written under the author's direction.

The basic philosophy of the Colombian geometry curriculum is the active exploration of space and of its modes of representation, both internal: in imagination and conceptualization, and external: in figures and movements.

When dynamic geometry, or active geometry, is mentioned, there are two typical misinterpretations: either it is understood as transformation geometry via reflections, or it is assumed that one starts with rubber sheet geometry (topology), and climbs up the ladder of the *Erlangen Program* through affine and projective geometry to Euclidean geometry, via transformation groups. The reader is asked to refrain from such possible interpretations.

1. School curricula in Colombia

School curricula in Colombia were centralized and compulsory up to 1994. The nine-year basic education consists of five primary (1-5) and four secondary grades (6-9). The last curriculum proposal started to be designed for grades 1 to 3 in 1976 and was tried out in a few experimental classrooms. A new version for grades 1 to 5 was designed under the author's direction from 1978 to 1981, and was discussed by the academic community and field-tested for three years. A third revised version, reported in 5 volumes from *Ministerio de Educación Nacional de la República de Colombia* published in the period 1983–1986, went into effect as compulsory syllabus for grade 1 in 1985 and - adding a new grade each year - ended in grade 5 in 1989. The syllabi for grades 6-9 came out gradually from 1988 to 1991 in 4 volumes published by *Ministerio de Educación Nacional de la República de Colombia*; they were generalized in 1993.

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Since 1994 all schools in the country are allowed to construct their own local curriculum following very general guidelines. The syllabi issued by the Ministry of Education are therefore no longer compulsory, but only provide the basic material for the development of each local curriculum.

2. The notion of mathematical systems to play games

The basic conceptual tool of the Colombian syllabi for the mathematics curriculum in grades 1-9 is a version of a General Systems Theory adapted for the teaching of mathematics by the author in the period 1978-80. According to this approach, each mathematical subject is shaped into a specific system with three well defined collections of ingredients: a basic collection of elements, a basic set of operations or transformations, and a basic set of relations (Vasco, [1], [2]).

To give an example, a system in plane geometry for grade 2 may consist of the set of undirected line segments (which form the basic collection of objects), the set of all parallel translations and all rotations (which form the basic collection of transformations), and the following six relations: incidence, parallelism, perpendicularity, length-equality, and the two usual strict length-inequalities (which form the basic collection of relations). With this system, pupils may play one or more 'games of strokes'.

Notice that, in this geometric system, parallelism of segments cannot be defined by non-incidence, and that perpendicularity does not require incidence; notice also that points and planes are not necessarily elements of this system, but could easily be added to new systems.

Every geometric system has three levels: a basic level at the concrete world of games, activities, images, ideas and language familiar to pupils; a second core level of conceptual games; and a third superficial level of symbolic games.

The general methodology is not to proceed, as usual, from the symbolic level to the conceptual, and then to concrete examples and illustrations, but from the concrete level to the conceptual, and only then to the symbolic level, at first with words, figures and symbols invented by pupils, and then with the words and symbols of the general mathematical community.

For each mathematical subject and each grade, the curriculum offers one or two examples of systems and ways to play games, but teachers and pupils can choose many others within the same subject. Pupils should be encouraged to invent names for objects, operations and relations, and to change the rules or specify goals for different games with the same system.

3. The distinction between geometric and metric systems

For curricular design in geometry it is important to distinguish between geometric and metric notions, especially in Colombia, where metric notions are

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considered to be constructed later than geometric notions. (Here, 'metric' is used as 'related to numerical measurements', and not as the widespread usage in the Anglo-Saxon countries: 'related to decimal systems of measurement').

Geometric notions are considered prior to metric notions, not because geometry initially is just topology, but because Euclidean geometry can be developed as a pre-metric subject, i.e. without any specific non-negative real-valued function on pairs of points satisfying the conditions of a metric.

Obviously, an underlying metric is needed to distinguish straight lines from curved lines, and shorter line segments from longer ones; but a specific numerical realization of such a metric is not necessary to do geometry. The intuitive notions of length, area, volume and angle-measure as specific physical magnitudes can and should be reconstructed without numerical assignments through pre-metric games; only later specific measurements within a given system of units need to be calculated to play metric games.

The distinction between pre-metric geometry and metric geometry is therefore not to be compared to Klein's hierarchy, but to a different type of distinction: between the non-numerical quantitative and the numerical quantitative aspects. (It is not a distinction between qualitative and quantitative aspects, as all of them are quantitative from the beginning.)

4. Dynamic geometry is not "Erlangen Program" geometry

The new Colombian curriculum starts exploring the intuitively Euclidean space of everyday experience and the properties of solids and plane figures via the usual even isometries (which we call 'sliding motions': translations, rotations and combinations thereof). Then the odd isometries (reflections) are studied, and then we explore only those affinities that are rescalings (homotheties with a fixed center and a fixed rescaling factor).

There is no pretence to impose on children a more or less disguised form of the Kleinian program in grades 1-9; the only attempt which could be viewed as exploring this Kleinian direction is the construction (not the definition) of the group concept through games of motions of cut-out figures in grade 8.

5. Dynamic geometry is not the usual transformation geometry

The importance given in the Colombian curriculum to actively translate and rotate objects, to reflect them via drawings, mirrors, transparent plastic sheets and imagination, should not be a cue to confusing dynamic geometry with standard transformation geometry as described for example in the books by A.F. Coxford and Z.P. Usiskin: *Geometry: A transformation approach*, published by Laidlaw/Doubleday in 1975 and H.R. Jacobs: *Geometry*, published by W.H. Freeman in 1974 (compare Vasco, 1992). One

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could study reflections only, and then through them study all plane isometries. But true reflections are not realizable through concrete materials. They occur only in the imagination. Either you reflect in your brain, or you don't reflect at all. The objective is not to explore all plane symmetries; it is to explore space and its representations, among them drawn or painted sketches of solids or plane figures.

Motions, like sliding a block or rotating a ruler, are the best starting points for geometry; rotations and parallel slidings need not be defined in terms of any other primitive idea. They can be left as undefined terms, since they are perceived directly (they are "body-syntonic" in Seymour Papert's terminology; see the book by S. Papert: *Mindstorms: Computers and powerful ideas*, published by Basic Books in 1980, pp. 58 and 63). Those two types of motions can be used by a child to construct its internal geometric space, but the corresponding groups need not be direct objects of reflection until after the 9-year basic education. The Colombian curriculum does not advocate transformation geometry, but only that geometry be introduced through those games which involve motions that help explore space and its representations.

6. Overview of the distribution of topics

Grades 1-3:

- Spatial relations; solid, hollow, rigid, flexible bodies; shapes, surfaces, edges, corners.
- Games of motions, search for regularities under motions.
- Surfaces as sides of solid bodies; flat, curved, round surfaces; bounded and unbounded plane regions.
- Lines as edges of bodies, surfaces or plane regions; types of lines.
- Points as corners, vertices or end points of line segments.
- Games of sliding and rotating lines; angle of rotation.
- Plane line figures; classification of triangles, quadrilaterals, etc.
- Games of sliding and rotating plane (lines and regions); active and passive symmetries; properties of figures through regularities under motions.
- Metric games with straight line segments via arbitrary units (bodycentered, object centered) and international units (m, cm).
- Metric games of measuring angles via rotations.

Grades 4-6:

- Models of solids; grids and nets; drawings of solids.
- Plane line figures; classification of bilaterals, trilaterals, quadrilaterals, quintilaterals and sexilaterals; general line figures.

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- Plane regions; classification of monogons, digons, trigons, tetragons, pentagons and hexagons; general regions.
- Drawings and constructions with rulers, compasses, triangles and other drafting instruments.
- Relations among figures and lines within a figure; short informal proofs and deductive chains.
- Translations and rotations; general sliding motions; reflections; general motions of figures in the plane; active and passive symmetries; congruence; art and symmetries.
- Introduction to Pythagoras' theorem; intuitive proofs; special cases: isosceles triangle, $30° - 60° - 90°$ triangle; 3-4-5 and other Pythagorean triplets.
- Metric games of lengths of straight and curved lines.
- Metric games of areas of squares and rectangles via counting squares; shortcuts to counting squares (area formulas).
- Metric games of angles in fractions of turn and degrees.

Grades 7-9:

- Translations and rotations; general sliding motions; reflections; general motions of figures in the plane; active and passive symmetries; congruence.
- Rescaling of figures; global changes of scale in drawings; combining rescalings with rigid motions: homotheties.
- Active and passive symmetries; similitude of figures; art and symmetries.
- Direct and inverse motions; combining two, three or more motions; identity motion; properties of compositions of motions.
- Study of general polygons; short deductive chains; Pythagoras' theorem; intuitive proofs; study of proofs.
- Definitions and undefined terms; theorems and axioms; rules of inference; deductive chains.
- Games of changing scales along only one axis.
- Coordinate axes; Cartesian and polar coordinates; changing scales in coordinate systems; ratio and proportion.
- Linear functions, representations of linear functions via tables, bar graphs, line graphs, Cartesian coordinate graphs, machines, lists of instructions, algebra.
- Drawing three-dimensional objects in two dimensions; projections, perspectives, multiple views, planar sections of solids, conic sections.
- Metric games of volumes and capacities; volume of cubes and paralellepipeds via counting cubes; shortcuts to counting cubes (volume formulas).

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- Metric games of lengths, areas and volumes in other systems of units; conversion of units.
- Metric games of angles in other systems of units; radians; conversion of units.

7. Technology and the geometry curriculum

Because of economic and cultural reasons, no mention is made in the syllabi issued by the Ministry of Education about computers. Most public schools in the country have no computers, or just one or two for administrative purposes. The link to technology is made via technical drafting, which prepares for the use of paint, draw or CAD software packets; above all, the whole dynamic geometry approach prepares pupils for computer implementations much better than the traditional static geometry approach, as private schools with good computers have already found.

8. Conclusion

The Colombian dynamic geometry curriculum invites teachers and pupils to ride with Thales and Pythagoras, Euclid and Proclus, Hilbert and Bourbaki, but as an active exercise of body, arms and hands, in imagination and in drawing; the main stress is in never forgetting the dynamic, active character of geometry, which is lost if a blackboard is filled with dead figures and static symbols. That cannot be geometry: only the corpse of geometry.

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Geometry in Egypt

Ghada K. Gholam

The educational system in Egypt is to a large extent centralised. It is the Ministry of Education, along with its subordinate boards, that is responsible for education at all levels. The Ministry of Education determines curricula and teaching methods for schools of all types – government/public

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and private – all over the country. The language of instruction is Arabic in the state schools and in some private schools. In foreign-language public and private schools, the language of instruction is either French, English or German. The foreign-language public schools are called experimental schools. Education is compulsory from the age of six to fifteen, and public education is free at all levels.

The following report on mathematics education in Egypt, with special emphasis on the teaching of geometry, is based on the following sources:

[1] Private communication with authorities in Ministry of Education in Cairo.

[2] Ministry of Education mathematics textbooks for grades 1 to 11. Ministry of Education, Egypt. Different years of publication.

[3] Ebeid, W. (1992), *Mathematics education in the Arab States.* In Studies in Mathematics Education, Vol. 8, UNESCO publication.

Weekly Hours of Mathematics Per Grade:

Level	Grades	Math	hrs/week
Elementary	1 - 5	Geometry and Arithmetic	5
Preparatory	6 - 8	Algebra	2.5
		Geometry	2.5
	9	Algebra	2
		Geometry and Trigonometry	2
Secondary	10	Algebra	3
		Geometry and Trigonometry	2
	11	Algebra	1.5
		Calculus	1.5
		Mechanics	2

1. Curriculum

Geometry occupies an essential part of the mathematics curriculum in Egypt at all levels. It is taught together with arithmetic at the elementary levels, but is given a separate place in the mathematics curriculum at the preparatory and secondary levels. The above table shows the distribution of the geometry curriculum throughout the school system.

The elementary geometry curriculum involves description of different shapes; acquainting pupils with different types of curves; measurement of length, perimeter and area through counting whole units; types of angles; and simple constructions. The preparatory geometry curriculum is composed of plane Euclidean geometry with some formulas for area and volume.

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The secondary curriculum involves plane and solid Euclidean geometry and coordinate geometry with some emphasis on the role of proofs.

2. Textbooks

The curriculum is developed through a centralised board that defines and prescribes the syllabus. Two methods for adopting textbooks are in force: competition or appointment. The competition method has been used at all elementary levels for some time and is currently progressing into the preparatory levels. Textbooks are prepared by teams of professionals and enter into a competition for selection. The teams of authors include university staff, supervisors and senior teachers. For the secondary level, textbooks currently in use are those prepared in the eighties by teams directly appointed by the Ministry of Education.

All textbooks are primarily written in Arabic. For schools with foreign language of instruction, the textbooks are written in English, French or German. In some cases, mostly at the elementary level, these are direct translations of the Arabic texts. At the preparatory level, they are sometimes written in the foreign language at once by a team of Egyptian and foreign authors within the framework of the official prescribed syllabus. Here association with a foreign publishing house is needed. These procedures have been in effect since the beginning of the 1990's. There are no separate mathematics books for geometry. The mathematics books are composed of some chapters on arithmetic or algebra, depending on the level, and other chapters on geometry. A teacher's guide is provided for all textbooks.

Pupils can obtain, along with the textbook, a handbook containing supplementary problems and exercises for extra practice, to prepare pupils for examinations, especially for the General Secondary Certificate. In addition to that, a training manual is given free of charge for mathematics grade 8 pupils. These handbooks also contain examination-type questions. Some Egyptian mathematics educators object to this since they believe that the idea of teaching is to upgrade the pupils to high ability orders by letting them work with unfamiliar problems rather than training them to solve standard problems through repeated practice.

3. Examinations

Governmental examinations are compulsory for all schools at the end of every cycle: elementary, preparatory and secondary. Examinations are given in the language of instruction and are direct translations of the Arabic originals. At the secondary cycle separate examinations are given for the different subjects including geometry. For the preparatory cycle there are

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two separate examinations for algebra and for geometry, while at the elementary cycle there is only one combined examination for geometry and arithmetic.

At the end of the elementary level (grade 5) a screening examination is given at the regional level. At the end of the preparatory level (grade 8) the examination is given at the governmental level, while centralised national examinations are given at the end of the second and third secondary years (grades 10 and 11). These are screening examinations, called General Secondary Examinations (GSE) given uniformly throughout the country and marked by examiners appointed by the Ministry of Education.

The GSE comprise two levels of examinations, Math.1 at grade 10, and Math.2 at grade 11. Math.2 qualifies the pupil to study engineering and natural sciences. Mathematics is an optional subject for the GSE examinations, whereas Arabic, Religion and a foreign language are mandatory in both years. Furthermore the pupil who chooses Math.1 in the first examination is not obliged to chose Math.2 in the second year, and vice versa which indicates that one is not a continuation of the other. In general the pupil has four chances in two consecutive years with the flexibility to select the highest grade he obtains in any of these examinations. The examinations are given in May and August. The success level is around 80% at the preparatory level and 75% at the secondary level. Generally pupils get better marks in algebra than in geometry.

4. Teaching

Mathematics teachers hold a Bachelor's Degree in mathematics or mathematics education. In teaching they are bound to stick to the textbook and follow the program it lays out. However, they do have the possibility to present more applications of mathematics to outstanding pupils. Each teacher may use his own method of teaching within the scope and sequence of the curriculum. It is important to note that the geometry teacher is supposed to give special attention to each individual in the class to ensure assimilation of the concept, in particular when elementary school pupils are taught to use geometry tools and to measure. In general, however, mathematics teaching is plagued with home coaching (tutoring) due to the densely packed classrooms and the bad preparation of some teachers; a practice parents always complain about.

5. Problems in the Teaching of Geometry

According to the points of view of some supervisors in the Ministry of Education the difficulties in geometry lie in its teaching rather than in the subject itself. The main problem in the teaching of geometry is that pupils are not given enough chance to grasp the concepts at the elementary

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level. They are left with discrepancies that they carry with them to the preparatory cycle. This is further aggravated at the preparatory cycle where the teacher's concern becomes totally focused on the rate of success at the governorate examination. As a result, theorems are sometimes taught as facts and presented in an elementary way without probing their concepts and applications. Therefore pupils do not know how to apply them. The Ministry of Education is doing its best to improve the situation by in-service training of teachers and close supervision. As in most countries, the geometry curriculum in Egypt is mainly classical (synthetic) Euclidean geometry, which seems to cause difficulties with teaching all over the world. Analytic geometry is taught at the secondary school level. Vector geometry is also taught at the secondary school level but as an optional subject depending on whether pupils study science or liberal arts.

6. Future Planning

The responsible people in the Ministry of Education suggest that in the future planning of mathematics one should consider the idea of having general mathematics at the preparatory cycle, i.e. presenting geometry and algebra in such a way that each algebra unit is followed by a geometry unit using algebra concepts studied in the previous unit. This will eliminate basic separation of geometry and algebra. The idea is still under consideration. If carried out, it might serve to give the basic concepts in a simple way helping the pupils to understand and appreciate mathematics. The Ministry of Education is also planning to improve the new mathematics books by introducing the basic concepts by appealing to daily life experiences. The aim being to integrate mathematics with other school subjects, and to upgrade the program according to international trends in mathematics education.

So far no use of new technology is applied in geometry teaching. There are computers in some schools but they are not used for teaching geometry or any other subject. They are used for special activities, and not as teaching aids. Future plans will consider introduction of teaching that is assisted by new technology.

Geometry in the Polish school: present state and perspectives

Jerzy Tocki and Stefan Turnau

Since Poland revived in 1918 it has had a curriculum compulsory for all schools. In 1989, when democracy was restored after the fall of the communist regime, this rule was abandoned. Now, in principle, teachers, schools and other bodies may legally propose, get approved (which is easy) and implement their own curricula. So hundreds of curricula are functioning in

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the country, though it is rather structure than contents that varies. The same concerns textbooks: formerly only one state-approved textbook for each age, now dozens. In the present circumstances it is therefore not an easy task to describe the situation concerning school geometry in Poland.

The compulsory elementary school starts at age 7 and lasts for 8 years until 15 years of age. Afterwards the pupils can choose between a 3, 4 or 5 year secondary school. In most cases, however, they will have to pass a written entrance examination in mathematics and national language. This examination has a strong influence on the contents and the pedagogy of mathematics in the last two or three years of mathematics in elementary school. Since the abilities to do calculations, to draw figures and to solve standard problems are the most easy to evaluate in written tests, only such things are put in the examination papers, and consequently mainly these skills are taught in elementary school.

The geometry contents in the 8-year elementary school include: basic notions and facts of Euclidean plane geometry, basic isometric and similarity transformations, trigonometry of right triangles, calculations of areas and volumes of standard shapes and bodies. In the secondary school, since global deductive organization of the course was abandoned in 1983, it has become a loosely connected set of classical Euclidean definitions and facts (rarely proved), with some use of transformations, vectors and coordinate algebra. The official curriculum advocates fusion of 2 and 3-dimensional geometry; but in the textbooks this often amounts to rather arbitrarily putting topics next to each other and, accordingly, one would expect the same to happen in the classroom. In the final examination (baccalaureate), for those pupils who complete a 4 or 5 year secondary school, mathematics is optional; and in the examination papers a great majority of questions in geometry require only calculation of areas or volumes, never proving anything, let alone generalizing or discovering things. This type of examination questions again more or less imposes the contents and the way of teaching on the teacher.

There are schools and classes where geometry is taught differently: new topics are presented, open approach is favoured, technology is used, etc; those are very rare. It depends mainly on the teacher, his competence and motivation; unfortunately, both have run short.

Schools are being equipped with computers and quite a number of secondary schools now run a computer laboratory. But this is used mainly for the teaching of computer science (file management and programming) and very rarely for mathematics. The Cabri software, fully translated and distributed in Poland, is used by less than 100 enthusiasts. In this case again the reason is lack of competence and motivation of the teachers.

The perspectives for an improvement of geometry in the schools are poor. The tendency to reduce the number of mathematics classes on the one hand, and on the other hand the strong emphasis on symbolic manipulations as

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the principal prerequisite for mathematics at a higher level, will certainly rather reduce than enhance the amount and importance of geometry in the curriculum. The only hope is that the enthusiasts will influence others with their enthusiasm and thereby initiate a slow process of change.

Geometry in China: the teaching material

Tang Shengchang and Zhang Fusheng

Except for a brief period 1950–80, the teaching of geometry in China has been uniform throughout the country. About 200 million primary school and middle school pupils were exposed to the same curriculum, the same syllabuses and the same teaching materials. All courses taught were compulsory and each piece of teaching material mandatory, and there were no voluntary courses or optional teaching material. Teaching of geometry in the compulsory courses was almost exclusively based on traditional Euclidean geometry. For many decades, there were only a few changes even concerning details and no thorough changes in the system with respect to the instruction or the level of the courses.

1. Ways of compiling teaching material for the curriculum

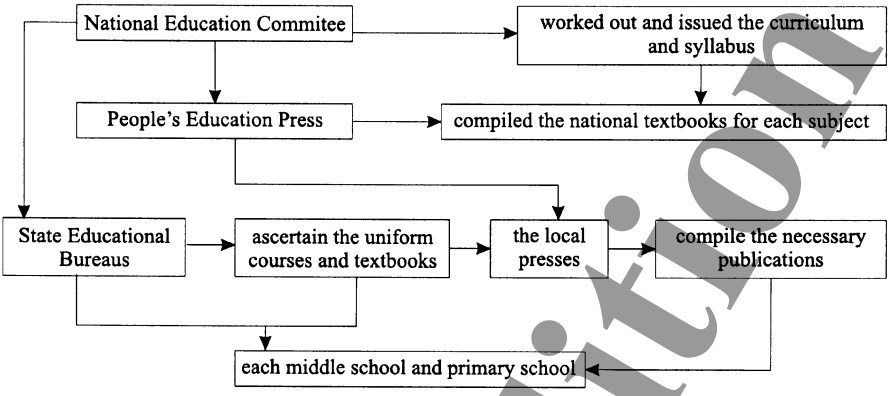
Like all the courses in any subject taught in the schools in China, the ways for compiling material for the geometry courses have undergone two stages.

1a. From the 1950's to the end of the 1980's.
During this period:

- The curriculum and the syllabus of each subject in the Chinese schools were worked out and issued by the State Educational Administrative Department - formerly the Educational Department, now the State Education Committee.
- The teaching materials in the Chinese primary and middle schools were compiled and published by People's Education Press run by the Education Department. Other publishing companies could only compile and publish the necessary collection of exercises, hanging charts and so on.
- The local educational administrative departments - education bureaus - were only in charge of the implementation of curriculum and syllabuses, the administration of teaching materials and the quality of the teaching. They could in addition also compile some local supplementary teaching materials.

The following diagram shows the system of compiling the material for the curriculum in primary and middle schools in China in the period from the 1950's to the end of the 1980's:

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1b. From the end of the 1980's to the present.

In progress with the reform expansion, the administrative system for the compilation of teaching materials has shown a tendency of diversification. A new policy has been installed, which imposes a uniform curriculum but possibly different teaching materials. This policy permits local authorities and individuals to compile teaching materials which follow the uniform curriculum but fit the needs of the different areas and the different levels of the pupils.

During this period:

- From 1988 onwards, in addition to the publications from the People's Education Press, alternative teaching materials (following the uniform curriculum) were published by Guangdong Educational Press, Sichuan Educational Press, Hebei Educational Press and Biejing Normal University Educational Press.
- The overall reform occurred in Shanghai and Zhejiang both in curriculum and teaching materials for each single subject. In fact, these regions have different curriculum plans as well as different textbooks.

2. Contents of geometry in lower grades (5–9)

We can only indicate the main topics common to the Shanghai curriculum and the national curriculum. In broad outline they are:

- Line segments, angles and their measurement
- Angles in relation to intersecting lines and parallelity of lines
- The various types of triangles and congruence of triangles
- Various types of polygons
- Similarity of figures

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- Computations with right triangles
- Circles, tangents to circles and angles in circles

3. Contents of geometry in upper grades (10–12)

For the teaching of geometry in the upper grades in the schools in China, the main topics are:

- Plane vectors, plane curves and their equations
- Analytic geometry in the plane
- Conic sections
- Straight lines and planes
- Vectors in 3-dimensional space
- Computation of distances and angles between lines and planes in 3-dimensional space

4. The role of proof in the teaching of geometry in China

Most experts of the mathematics education circles in China think that in the teaching of geometry, and especially in the teaching of plane geometry, pupils should not only be taught Euclidean geometry as a collection of facts, but should also learn to prove theorems. Studying proofs in geometry is very good for the pupils in order to master the mathematical methods and to develop the ability to think, and hence this should be an integrated part of mathematics teaching at middle schools.

At the age of 13–14 (grade 7 or 8), Chinese pupils start to learn to do proofs in geometry. In order to have children of this age succeed with such a difficult task, many necessary reforms had to be done in all kinds of teaching materials. The guidelines for the reforms have been:

- Use the main ideas of Euclidean geometry, and change the traditional methods of explanations.
- Go from objects to the abstract; from giving reasons to giving proofs; have the pupils master the methods of proofs in geometry.
- Through reasonable arrangement of assignments, help the pupils grasp the ideas formed; at the same time, allow them to deal with the facts of geometry by using vectors and coordinates as appropriate.
- Do not introduce the axiomatic construction of the whole of Euclidean geometry to the pupils.

It has turned out that more than 90% of the pupils can learn to do proofs in geometry if these guidelines are followed.

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The geometry curriculum in Germany: past and future trends

Michael Neubrand

To describe evolution and status of geometry curricula in Germany is more complicated than in other countries. Since educational policy in Germany is a matter for the States (Bundesländer) and not the Federal Administration, we essentially have 16 more or less different syllabi, one for each State, and in each State 3 or 4 syllabi for the different school types at the secondary level. Probably even more influential is that the traditions of teaching in the respective school systems differ among the States, e.g. in the roles the different school types see themselves, in the system of examinations, and also in teacher education. Curricula are defined and revised by committees of the single States which work independently of each other, although there are some basic conditions to be fulfilled in order to ensure a certain degree of compatibility and comparability of the curricula within the Federal Republic. Cf. Schupp et.al [2] for a more detailed report on mathematics education in Germany. Nevertheless, there are some common features in geometry which may be seen as a kernel of the content of geometry teaching in Germany.

Primary Grades (pupils from 6 to 10 years)

At the general level, geometry aims in these grades at making the pupils conscious of their spatial experiences. To that end, a lot of different tasks were proposed, and a variety of material has been developed and can be used, like: making and analysing models of solids and shapes; paper folding and other constructive tasks; drawing and interpreting different kinds of maps; exercises for visual thinking; etc. Also, first steps into a deeper and more abstract analysis of plane and spatial figures are done; e.g. an often-taught topic is to find all possible patterns of an unfolded cube. There is a vigorous discussion in teacher journals and books about how to teach geometry in primary grades; but there are also some complaints that, in everyday teaching in schools, geometry does not play the role it deserves from the standpoint of developing mathematical attitudes and understanding.

Explicit and conscious geometry teaching in primary grades is a relatively new topic. It started as a part of the New Math movement in the 1960's and early 1970's. But in contrast to the disappearance of the arguments for arithmetic, the reasons given then for including geometry into the curriculum are still believed to be convincing. An explanation for this may be that these reasons were based on a sound discussion of both the potential of the mathematical content and the psychological and educational implications.

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Lower Secondary Grades 5 - 10 (pupils from 11 to 16 years)
Geometry always was, and still is, a classical subject in these grades. But in the different school types emphasis is given to different aspects of geometry. In Hauptschule (= more practically-oriented basic school, preparing for vocational training) geometry is considered mainly as a means for mastering practical tasks including measurement. In Realschule (= middle school, preparing for vocations in industry and trade) the focus is on geometry as an important part of knowledge for engineering, designing and constructing. The classical Euclidean style of geometry appears in the Gymnasium (= more theoretically-oriented school which provides pupils, after completing also the upper secondary level, with the entrance qualification for the universities); geometry is seen there as a subject dealing with figures and shapes, but employing logical inferences to come to deeper insights into the matter.

In broad outline, the distribution of the most important topics of geometry in the Lower Secondary Grades is as follows:

Grades 5-6: Basic exercises in drawing with straightedge, compass and protractor; basic concepts like lines, parallels, angles, circles, from an empirical point of view; recognition of basic symmetries.

Grades 7-8: First steps into reasoning about geometrical patterns, either on the basis of using geometrical transformations or with the method of congruent triangles; essentials of Euclidean congruence geometry; measurement of simple areas and volumes.

Grades 9-10: Measuring lengths, areas and volumes of more complicated figures; Pythagoras' theorem; area of the circle and the sphere; similarity; basic trigonometry.

Upper Secondary Grades 11 - 12 or 13 (pupils aged 17-18-19)
In these grades which mainly are at the Gymnasium, but to a smaller extent also at some college-like schools which give alternative ways to the entrance into tertiary education, geometry appears mostly as Analytic Geometry. It depends on syllabi, teachers and also on the choice of the pupils, whether geometry is seen closer to Linear Algebra, or closer to geometrical topics like conic sections, algebraic curves, elements of spherical geometry, or even sometimes elements of non-Euclidean geometry.

Geometry teaching in Germany traditionally had a considerably high level, at least considered from the point of view of the content; another question is, of course, what really was learned and realized by the pupils from that fairly advanced body of knowledge. In the New Math period, there was then a decline of geometry teaching in schools since the logical basis of geometry is indeed very complicated. So in the last 20 years, geometry had to defend its position and to find a new interpretation of its role. These considerations

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opened the mind for the many aspects of geometry (Vollrath [3]) and called for wider perspectives which should rule the educational debates about geometry in schools.

The general tendency today is therefore directed towards a kind of multi-perspectivity as the appropriate basis for geometry teaching (cf. the extensive report of Graumann *et al.* [1]). These multiple perspectives include e.g.: the acknowledgement of informal pathways to geometry in the lessons; the shift from applications to the process of modelling; conscious reflexions about working mathematically using geometry as a generic example; a more pragmatic view of the formerly ideological debates whether geometry is better based on transformations than on the Euclidean method of congruent triangles; a less formal view on proving, in favour of an active access to mathematical theorems; the still ongoing process of inclusion of new technologies into the curriculum.

The reasons for this awareness of the multi-perspectivity of teaching geometry lie in changed views on mathematics itself (from formalism alone towards a broader view of mathematics as a human activity) and on teaching and learning mathematics (towards more emphasis on the active roles of the pupils). Therefore, the recent discussions on the future development of the curricula in geometry concentrate on issues like: fostering active learning by the methods of working in groups or project-oriented teaching and learning; overcoming the traditonal separations of the different school subjects by integrating and interpreting geometrical topics in contexts from the natural, technical or social sciences.

Computers play a crucial role in this process of integration. They should not be limited only to the tool function; rather, used with interactive geometry software, like Cabri, or Thales (a German-Austrian analogue), computers may allow the inclusion of new topics into the geometry curriculum, or teaching well-known topics in an alternative way. But discussions about the actual implementation in classrooms and the critical evaluation of the educational effects of the computers in the geometry curriculum are still continuing, despite the fact that many proposals for the use of geometry software have already been made.

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Section III. LOOKING TOWARDS THE 21ST CENTURY: POSSIBLE TRENDS IN GEOMETRY CURRICULA

Vagn Lundsgaard Hansen

If we look ahead to a not too distant future, we can perhaps extrapolate from the present trends, and guess that in several countries a large proportion of the population will go to university - which will become the common place for a tertiary education. Morover, many adults with university studies behind them will return to university from time to time to be retrained in some distinct kind of work in connection with their profession. In such countries university thereby will become a place for a second or third training.

In these circumstances the distinction between *mathematics for all* and *mathematics for those who will go to university* might somehow lose its meaning. But a distinction still has to be made between those who will attend science and technology faculties and those who will attend all other faculties. When selecting the geometrical content in the curriculum at the secondary school level, it will therefore become increasingly important to choose such units of geometry which foster the right skills, abilities and attitudes for a meaningful and useful tertiary education.

In countries where education for the majority of the population at the moment ends in early grades, it is to be hoped that in a near future it will be possible to extend mass education at least until middle schools. It is still practicable to recommend that curricula in mathematics are structured in such a way that geometry occupies a significant place. The guidelines developed earlier in this chapter might serve as inspiration, avoiding too much abstraction and too much formalism, and linking geometry to local culture and tradition, not just adopting curricula from some leading country.

A big problem in school will probably continue to be a point of discussion, namely from which grade on should there be different curricula reflecting the future choices of the pupils. When it comes to geometry, it can conveniently take place somewhere between grade 8 and grade 10 in school. For those pupils who have more interest in mathematics, the conceptual and structural aspects of geometry should begin to be stressed. For those who are less motivated for mathematics, it is advisable to stress more the applications of geometry, like perspective, coordinate geometry and the ability to draw and to interpret visual representations of statistical data, without any attempt to teach them axiomatics or abstract vector spaces and the like. Moreover, there may be countries where mathematics – in particular geometry – is not compulsory from a certain grade on.

In the 21st century computers will be part of daily life, and teaching at all levels might to a large extent be based on interactive computer programs. This will probably be more to the benefit of those who are already

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motivated and successful, than to the students who have difficulties with the traditional ways of teaching. It will be an important task to rethink the role of teachers and to find alternatives for personal contacts in such an educational environment. In these didactical considerations one should not exclude the possibility that the changes might not go very much further than making available a very powerful teaching aid, like a "dynamic overhead". Under all circumstances the understanding of concepts and the ability to do creative thinking with these concepts using all available tools will be increasingly important. Since computer programs make strong use of visualization, and visual thinking must be learned, we predict that the interest in basic geometry will grow.

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8. ASSESSMENT IN GEOMETRY

This chapter consists of two sections. The first, by Mogens Niss, looks at assessment in a general setting, before discussing some possible detailed types of assessment. In section II John Pegg, Angel Gutiérrez & Pedro Huerta describe a type of assessment derived from research on the SOLO Taxonomy that extends the van Hiele "levels".

Section I. DIMENSIONS OF GEOMETRY AND ASSESSMENT

Mogens Niss

INTRODUCTION

The term "assessment" in mathematics refers to the identification and appraisal of students' knowledge, insight, understanding, skills, achievement, performance, and capability in mathematics. Assessment does not concern the judging of educational or instructional systems or programmes, curricula, teachers' competence, teacher training, etc. For such activity 'evaluation' has become the standard term (see, e.g. Niss, [7]). Evaluation will not be considered in this chapter.

The constellation "geometry and assessment" relates to two quite different issues. The first issue may be given the following condensed and over-simplified formulation: "what can assessment do for geometry?". This formulation covers questions such as: "in the teaching and learning of geometry, what is the actual and potential role of assessment?", "what specific assessment problems manifest themselves with respect to geometry?", "what modes of assessment are particularly well-suited (respectively ill-suited) as regards teaching and learning of geometry?", "what aspects of learners' geometric knowledge, insight, and skills are traditionally assessed in the classroom, at tests or at examinations?", and so forth. In other words, here the focus is on geometry as an educational topic, and assessment occurs as a specific set of glasses through which this topic is being viewed at the moment.

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The second issue is the converse: "what can geometry do for assessment?". In this context the focus is on assessment in mathematics, and geometry is considered a (potential) vehicle for the pursuit of assessment interests. For instance, it may happen that geometry possesses specific properties or qualities that make it especially helpful for the assessment of certain aspects of mathematical knowledge, insight and skills.

Both issues are interesting and deserve attention, but they should not be mixed up, let alone be confused. Since the present book deals with the teaching and learning of geometry, it is primarily the former issue that should, and will, preoccupy us in this context. However, as we are able to argue that geometry does offer valuable and special opportunities for the assessment of mathematics in general, we shall in fact consider the latter issue as well, albeit fairly briefly.

1. The role of assessment in geometry

Basically, there are three general purposes of assessment which pertain to assessment in mathematics as well. The fundamental one is the provision of *information* – whether to the individual learner, to the teacher, or to the educational 'system' in which the learner is situated – about the learner's performance, achievements and capabilities. Such information is typically designed to serve a further (ultimate) purpose, namely to make *decisions* and to take *actions* regarding the future of the learner. Some decisions or actions will be on the part of the learner, with respect to matters such as controlling his own learning strategies and activities, or choosing and preparing for future education or career, while others implicate the teacher or the institution at issue and deal with pupils' and students' passing/non-passing of tests or exams, and the filtering, selection and placement of each learner; whereas still others involve the education system at large in relation to pupils' and students' obtaining a licence or a certification, qualifying for various sorts of jobs etc. The final purpose regards the *shaping of reality.* This is divided into two parts, the reality of teaching, and social reality in a broad sense. Firstly, it is a classical purpose of assessment to control, at least partly, the content, the format, and the outcome of teaching and learning, including teachers' and learners' activity and behaviour. In this respect assessment serves as an instrument for educational authorities to control, perhaps on behalf of society, the agents of the classroom. Proceeding along this line we come to the second part (which is often more a tacit and indirect than an explicit purpose of assessment): the shaping of social reality in the school/institution, in the education system, in the labour market, or in society at large, brought about by the way in which assessment induces norms, attitudes and behaviour in the social environment.

As specifically regards assessment in geometry, the provision of information to the individual pupil or student, and to his teacher(s), comes to

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the fore as the more significant purpose, together with the purpose of establishing a basis for such decision-making or action-taking that affect the learner's controlling of his own learning behaviour in geometry or his choice of future educational path. In some cases also the purpose of exerting an influence on teachers' teaching and learners' learning is put on the agenda by curriculum authorities. The remaining purposes are hardly connected to geometry in particular but rather to mathematics (and education) in general.

Now, what types of information about a pupil's or a student's performance, achievements, and capabilities in the area of geometry may be wanted or needed? The answer to that question depends strongly on the goals (ranging from ultimate ends, over general aims, to concrete objectives) which are set for the teaching and learning of geometry, for it is primarily the attainment of these goals about which information is sought by means of assessment. Furthermore, once it has been settled what information is to be requested, the issue emerges, of choosing or designing assessment modes which are compatible with, and appropriate for the goals involved and the types of information demanded. In the sections that follow we shall deal with the goals pursued, the information sought, and the assessment modes adopted, all in relation to geometry.

2. The goals of geometric teaching and learning

The goals for teaching and learning geometry fall into three categories which are, then, reflected in the information that assessment is meant to provide.

The first category consists of goals related to the acquisition of knowledge, insight, and skills attached directly to the geometric subject matter belonging to the topic(s) taught in the context under consideration, whether this topic is synthetic Euclidean or non-Euclidean geometry, analytic, projective, algebraic, differential or other type of geometry. Here, the goals typically regard the acquisition of knowledge about such notions, concepts, definitions, methods and theorems – and the skills needed for dealing with these elements – that are characteristic of the topic at issue; understanding of the foundation and theoretical architecture of the topic; and insight into its interrelations to other areas of geometry in particular and of mathematics in general. Goals of this category may also, however, regard knowledge of the historical genesis and development of the topic, or appreciation of philosophical (or psychological) issues related to its foundation, interpretation or application.

The second category of goals address geometry as a unique instance of intersection between mathematics as a pure, abstract and general theoretical edifice, on the one hand, and physical space, objects and phenomena in nature or in the man-made world, on the other hand. Put differently, these goals concern what we have called, elsewhere in this book, the dual nature of geometry, i.e. geometry as both a branch of mathematics (or

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several branches if you like) as well as a natural science and a 'design science'. Goals which may be relevant in this context range from the very understanding of this dual nature of geometry, over the ability to understand and activate the interplay between geometric topics, viewed as pure mathematics, and geometric aspects of 'natural or artifactual nature', to the ability to build, analyse and utilise specific geometric models (in the modern sense, not in the old sense of plaster or wood-and-wire models of abstract geometric objects) in concrete application situations.

The goals of these two categories are specific to geometry and cannot be equivalently replaced by goals connected with other branches of mathematics.

The third and largest category of goals contains those which are special cases of goals related to the teaching and learning of mathematics at large, just restricted to the area of geometry. These goals occur in the context of geometry, either because geometry may be considered as a domain that is particularly well suited to pursue the goals in question, or simply because the teaching and learning of geometry, being on the agenda for independent reasons, happen to offer an opportunity to pursue mathematical goals of a general nature at the same time. In other words, in this category of goals, geometry acts on behalf of mathematics at large. This implies that a list of these goals would be almost identical to the list of goals that one could establish for the teaching and learning of mathematics in general, only specialised to geometry. Such a list would be very long indeed. So, we have to confine ourselves to indicating the most important sub-categories of general mathematical goals.

One group of goals focusses on learners' gaining knowledge of, and insight into, matters inherent in all mathematics: the nature and role of definitions, theorems and proofs; the relationship between general statements and special cases; the range and exemplification of concepts and propositions; the kinds of questions asked and the kinds of answers given in mathematics ("if-then" assertions; existence; "how many?"; uniqueness; equivalent characterisation; (exhaustive) classification; algorithmic computation; transformation; invariants; and so forth).

A second group of goals are concerned with the learner's knowledge and mastery of mathematical techniques and methods, such as: understanding, manipulation, syntactic and semantic interpretation of symbolic representations and expressions; rules of logical deduction; various general methods of proof and proving (computational, logical equivalence, direct, indirect, induction, etc.).

Another group of goals address puplis' and students' ability to pose, analyse, and solve mathematical problems, with respect to both routine exercises and open-ended, fuzzy, and perhaps intriguing, problems. Heuristic capabilities, creativity, and inventiveness are closely linked to this group of goals and to the following one as well.

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A group of goals address learners' ability to activate and apply mathematics in extra-mathematical contexts in order to describe, understand, or master aspects of the situation under consideration, again both in routine and in non-routine contexts. This includes the ability to identify potential mathematical content and structure in the situation, to construct a mathematical model, to solve the mathematical problems formulated within the framework of the model, to interpret the outcomes in relation to the extra-mathematical context, to make inferences about this context, and, last but not least, to critically analyse and assess the model from mathematical as well as extra-mathematical points of view.

A fifth group of goals deal with learners' overall feel for and appreciation of mathematics as a discipline, its history and development, its philosophical nature and characteristics, its place and role in culture and society, its use and misuse, its relationships with other disciplines, etc.

The sixth group of goals pay attention to fostering, in learners, attitudes and beliefs about mathematics and about their own relationships with the subject and its exercise.

Finally, we may add a seventh group of goals which used to be in focus of mathematics education and which is still maintained as essential by many mathematics educators: formative education of the individual, i.e. the provision and training of general mental and personal capabilities to be put into practice in all sorts of context, also outside of mathematics. Traditionally, (axiomatic and deductive) geometry has been viewed as a pre-eminent training ground for exactly such education, even if, in modern times, considerable skepticism towards the justification of the underlying belief in its effectiveness has gained momentum.

3. Assessment of the achievement of goals

It is a fundamental purpose of assessment in geometry to generate information about the extent to which learners of a given category have achieved some subset of the goals outlined above. This subset typically varies with the type of learners considered. However, the goals involved are quite often only in the air, i.e. they are not made clear, let alone stated explicitly and articulately, neither by (or to) the assessor nor to the learner assessed. This fact is probably one – but not the sole – factor responsible for another fact: the traditional assessment modes adopted in mathematics tend not to be matched with the goals officially or semi-officially pursued in its teaching. And this lack of matching is not the least manifest in relation to geometry. It is a classical observation that, irrespective of the explicit or implicit goals underpinning the teaching of mathematics, the content and modes of assessment themselves give rise to immanent goals, namely those which can be derived from what it takes to succeed in assessment. Learners and teachers quickly begin to decode and see these goals as the real ones, and this has a decisive impact on teaching as well as on learning.

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Let us sum up, in condensed form, the gross set of kinds of possible goals for the teaching and learning of geometry, from which corresponding assessment goals are derived. These goals concern: geometric subject matter and theory; the dual nature of geometry; building and applying geometric models. And with respect to geometry on behalf of mathematics: characteristic features of objects, issues and statements in the discipline; ways of thinking, methods and techniques; problem posing and problem solving; creativity and inventiveness; the nature of modelling and applications; philosophy and history of the discipline in society and culture; learners' attitudes and beliefs; training of general mental capacities.

In order to avoid misunderstandings, it should be underlined that these goals are not (necessarily) advocated by the author of this section. Instead, they are established by analytic examination of the field.

4. Modes of assessment

However, let us assume, here, that the teaching of geometry serves some subset of goals such that each kind of goals is represented in the subset, and let us further assume that it is intended to attempt to assess, in a serious way, the achievement of the goals in the subset. Such an attempt requires the determination of the assessment modes to be adopted. By an assessment mode we understand a 'vector' that includes the following components (Niss [7], p 12): The *subject* of assessment (who is assessed?); the *objects* (what is assessed, in terms of subject matter content and learner ability?); the *items* (what sorts of output are assessed?); the *occasions* (when, in relation to the curriculum, does assessment take place?); the *procedures and circumstances* (what happens, and who does what?); the *judging and recording* (what is emphasised, and what is recorded?); the *reporting* (what is reported to whom?). A specific assessment mode is established through the specification of each of these generic components. Perhaps the most significant components giving rise to apparent differences between specific assessment modes are "the objects", "the items", and the "procedures and circumstances".

As to "objects", what is traditionally in focus is learners' knowledge of (geometric) facts (mainly names, definitions and properties of concepts); mastery of standard methods and techniques (elementary geometric operations and their combination (e.g. bisecting an angle by ruler and compass, computing quantities in a figure by means of trigonometry, determining the curvature of a curve in 3-space given in parametric form), proving that the diagonal of the unit square has irrational length); performance of standard applications (like finding the area of a piece of land with geometrically "tame" boundaries, taking measures that allow for the estimation of the height of a flag pole). It is much rarer to encounter assessment objects such as:

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– visualisation;
– open-ended problem solving;
– geometric modelling of complex, extra-mathematical situations;
– rigorous and heuristic reasoning;
– generation and exploration of hypotheses;
– explaining the structure of a geometric theory;
– establishing links between different geometric topics;
– interpreting an abstract geometric theory in relation to a specific object domain.

When it comes to "items" and "procedures and circumstances", traditional assessment primarily employs timed written tests or exams, in many places of a multiple choice type. Such tests are typically composed of several mutually unrelated closed-form exercises/problems that refer to different parts of the syllabus. Reproduction of definitions or proofs without the support of the textbook may be included as well. No matter whether we are talking about teaching-learning oriented, classroom based assessment (often called "formative") or about end-of-course ("summative") assessment, it is customary that pupils or students are required to 'sit the test' or the exam in a room in which complete silence rules and where communication with others is forbidden. At the end of the test session learners hand in what they have completed and their papers are scrutinised and marked, perhaps corrected too, by the teacher/assessor. Hence, the time constraint is a key component in the test. In addition to written tests, assessment may also make use of oral interviews with learners, in which learners may be asked to solve exercises/problems in front of the assessor, quote facts or recite definitions of concepts or proofs of theorems, or (more rarely) present or explain a topic or a piece of theory.

5. The need for innovation in the assessment of geometry

Presumably it seems quite clear that traditional assessment modes, based on components specified in the manner outlined in the preceding paragraphs, only allow for reliable and valid assessment of a fairly limited subset of the kinds of goals of teaching and learning of geometry which were listed above. More specifically, such assessment modes can deal with (aspects of) learners' knowledge of geometric subject matter and theory and mastery of corresponding basic techniques, their ability to solve closed or almost closed exercises and problems and to put (mostly standard) applications and models into use in not too complex situations. As far as the remaining types of goals are concerned – in particular those of a higher order, complex, comprehensive, and general nature – the modes traditionally adopted do not really allow for an adequate assessment of their achievement. For instance, suppose we agree (at least I do!) that we need to be able to assess learners' ability to translate and pass from one 'code' of, or perspective on, geometry to another, i.e. transfer from a visual situation to a verbal

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description and further on to one of several formal types of description (in, say, an axiomatic, algebraic, or vector space setting) and back again. By definition, assessment based on single-step tasks cannot capture transfers between different geometric codes and perspectives. To such and similar ends other assessment modes are needed, both in classroom practice and at the end of course or of a section of schooling. The remarkable thing is that such assessment modes actually do exist and are being/have been implemented in various places but mainly on a relatively small scale.

Some of these modes are based on items such as learners' written reports of larger projects, or of extended investigations. In addition to having their report as such assessed, pupils or students are sometimes assessed at an oral defense of the report as well. It may also be the case that instead of producing a written report, learners design a set of posters, an exhibition, a piece of computer software, a video programme, a teaching sequence, and so forth, which then form the item of assessment. In projects and extended investigations learners work, alone or in groups, for a longer period of time on complex themes, issues or problems. They may deal with the structure and organisation of geometric theory, the history and philosophy of geometry, geometric applications and modelling. Projects or investigations can be put into practice at all educational levels, but naturally the content, format and level of treatment will vary accordingly. At Roskilde University (Denmark) students of science and mathematics perform group projects (each occupying roughly half of the time in one semester) from the very beginning of their studies. To give just a few examples of projects on geometry: the geometry of map-making; the impact of the appearance of non-Euclidean geometry on the perception and development of mathematics; sphere packing in theory and applications; the history of angle trisection.

Other – less radical – forms of assessment are variations of the traditional, test type, ones. Here the main problem is that the time constraints inherent in most traditional assessment are so tight that they have to be disposed of, or circumvented, if more profound and complex geometric knowledge, understanding, insight, and skills are to be assessed. One way of circumventing these constraints is to give the learner a set of take-home questions that call for non-routine in-depth insight and capablity to be answered adequately. After a fair number of days, the learner hands in his paper which is then scrutinised and assessed by the teacher/assessor. Of course it is not possible to really prevent the learner from seeking advice or help from others before handing in the paper. This is no problem, however, as there is a second stage in which the learner has to defend his solutions in an oral interview with the teacher/assessor. At that interview, the learner is asked to explain the solutions adopted, and the theoretical bases of these, and to react to "what if instead?" questions, and so forth. It is mainly certain sorts of mathematical capabilities that can be assessed this way. For

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instance, the ability to find 'aha!-type' solutions to challenging problems – so abundant in geometry – cannot be appropriately assessed in this way if it is crucial to make sure that the learner found the solution all by himself. However, if this requirement is given up the format just described allows for both valuable learning activity and valid assessment.

Another assessment mode is the so-called two-stage test (see, e.g. de Lange, [3]). In the first stage, the pupil or student sits an ordinary timed written test. The resulting paper is scrutinised by the teacher, the most serious mistakes are indicated/corrected and the paper is returned to the learner. In the second stage, the learner is given a fair amount of time, e.g. two weeks, to revise the paper which is then handed in again and assessed.

As a simple example of an item which can be a component of a variety of realistic assessment modes, let us consider the following problem of applied geometry. Draw a curve – and then make a cut along it – on a flat rectangular piece of soft cardboard such that, if rolled into a cylinder with a circular orthogonal cross-section at its base, the cardboard will form a circular cylinder cut by an oblique plane. Although at first glance it may appear slightly exotic, this is in fact an authentic problem posed by an architect who wanted to construct a cardboard model of a building designed to have this shape. In a slightly more advanced form the problem also arises in plumbing if two cylindrical tubes (not necessarily of the same diameter) are to be fitted together at a certain angle. This problem can be approached in several different ways and at a variety of educational levels, ranging from a combination of guessing and physical experiment to exact computation in 3-dimensional descriptive geometry. The problem also allows for generalisation in different directions and thus may give rise to flexible learning and assessment activities for use in various contexts. This is just one example of an abundance of applied (authentic) geometric problems with similar properties.

The assessment modes briefly sketched here are certainly not meant to be exhaustive. Many other relevant modes exist and are described and discussed in recent literature on assessment (see, for example, Gravemeijer et al. [1], Kulm, [2], Leder, [4], Lesh & Lamon, [5], Niss, [9], Webb & Coxford, [11]). Most of all these modes are not specific to geometry, but they are highly relevant for geometry nevertheless. A closer investigation of the assessment of cognitive growth in geometric insight and understanding, based on the so-called van Hiele levels and the SOLO-taxonomy, is given in the next section of the present chapter.

The main point here is not to produce or discuss an inventory of suitable assessment modes but to emphasise that different aspects of geometric knowledge, understanding, insight, and skills need to be assessed through different means. There is no single mode which can meet all the requirements for rich, representative, valid and reliable assessment in geometry. What is needed is balanced collections of assessment modes which in total

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can cover the diverse kinds of major goals of the teaching and learning of geometry. It was indicated in the preceding paragraphs that there are, already, quite a few relevant assessment modes at our disposal for inclusion in such collections. It is mainly a matter of putting them into actual practice (cf. an example of this described in Stephens & Money, [10]). However, the obstacles for this to happen are many. Some of them reside in the politico-economic sphere (the 'alternative' modes are time consuming, resource-intensive and hence expensive, and are also met by conservative skepticism in society at large). Other obstacles can be encountered in the mathematics education community itself which, in addition to being fairly firmly rooted in the established assessment traditions, is not in general well acquainted with new developments in the philosophy, theory and practice of assessment. At any rate, in most places there is plenty of unexploited room for improvement in the assessment of geometry, and it is not the lack of reasonable assessment modes that is responsible for this situation.

Most of the examples put forward in the preceding sections concern the assessment of geometry at post-elementary educational levels. This is mainly due to the fact that at the primary level geometry is usually integrated with other mathematical or proto-mathematical activities rather than being constituted as a well-defined, let alone a separate, topic. Therefore the assessment of geometry at that level may tend to be inseparable from the assessment of mathematics at large. However, since the formation of geometrical experiences and knowledge ought to start already at a very young age, mathematical assessment at that level should encompass aspects of geometry as well, such as orientation in space; measurement; drawing; practical work with paper, cardboard, scissors, strings; and so forth.

6. The role of geometry in assessment

Up till now, we have primarily been dealing with assessment for the sake of geometry. In this concluding section we shall, quite briefly, consider the reverse issue, the use of geometry for the sake of assessment in mathematics. In other words, assuming, here, that the goal is to assess pupils or students in mathematics in general, does geometry have anything to contribute to this end? It seems that the answer is 'yes!'. We shall restrict ourselves to offering two arguments to support this answer.

Firstly, in the previous discussion of assessment referring to the goals of the teaching and learning of geometry, it was suggested that for a number of those goals geometry does, in fact, act on behalf of mathematics at large. To sum up, once again, we are talking of goals regarding the nature and the properties of basic elements of mathematical theories, mathematical ways of thinking, problem posing and solving, creativity and inventiveness, modelling and applications, history and philosophy, attitudes and beliefs, education of the mind. So, to the extent that the achievement of any of these goals is assessed in the context of geometry, it is also assessed from a

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wider mathematical perspective. This is not only the case because geometry is a branch of mathematics but also because so many mathematical concepts, objects, and theories possess geometric aspects, representations or interpretations. Let us just give one example: the relationships between proofs (also of theorems in, say, number theory), visualisation, and geometry, as elucidated so nicely in *Proofs without words* by Nelsen [6]. In these respects geometry certainly has something to contribute to assessment in mathematics, but so have other branches of mathematics. Nevertheless, it seems that because of the special status of geometry in mathematics and its special place at the crossroad between mathematics and physical reality, geometry offers a particular richness in issues, topics, situations, and problems that lend themselves to multi-faceted assessment of general mathematical knowledge, insights, and capabilities.

The second argument relates specifically to assessment in the area of problem posing and problem solving. Here, geometry occupies a fairly unique position (perhaps with the exception of probability and statistics, which have certain features in common with geometry but at a lower order of magnitude). The point is that there are infinitely many geometric problems (in one, two or three dimensions, primarily) which can be posed without reference to or invocation of any specific theoretical setting. Quite often such a problem may make sense in a realistic everyday (extra-mathematical) context (e.g. the design problem presented in the preceding section), and even if it does not, it may very well be formulated in such a way that a lay person can understand what it is all about. This has four important consequences.

(1) It becomes a substantial part of the solution process to specify the problem and give it a precise formulation that allows for mathematical (geometric) treatment.

(2) The problem can usually be viewed from different perspectives, be attacked by a multitude of different methods, and be imbedded in a variety of different frameworks of geometric theory.

(3) Because of (2), and partly (1), solution-approaches and solutions can be compared and discussed with respect to elementarity (degree of technical complexity of the solution method), simplicity (degree of ingenuity required for the solution), convincing power (degree of 'aha!-ness'), theoretical depth and generality (or particularity, for that matter).

(4) Issues such as the possibility of generalising the problem or the solution, while links to similar or related problems may arise naturally and inspire further exploration.

Let me quote just one example of a problem (non-trivial, I have to admit) which illustrates the points (1)-(4). In Fig. 1, which consists of three adjacent squares, what is the sum of the angles α, β, and γ?

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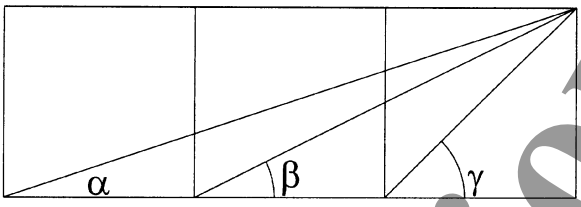


Figure 1

I thank Andrejs Dunkels (Luleå, Sweden) for having attracted my attention to this problem. The problem can be approached in numerous different ways (more than fifty, I was told), ranging from quite elementary solutions to fairly involved ones. As is always the case with good problems, the need to argue for the correctness of the solutions obtained gives rise to an excellent exercise in, and assessment of, proof and proving.

The arguments put forward above suggest that geometry does offer special opportunities for assessment in mathematics in general. Additional arguments could have been given on top of these, but let them suffice for the present context. Underlying all the arguments is the intrinsic and multidimensional role that geometry plays in mathematics as a topic and as an intimately intervowen aspect of so many other branches of mathematics.

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Section II. ASSESSING REASONING ABILITIES IN GEOMETRY

John Pegg, Angel Gutiérrez, Pedro Huerta

Introduction

Assessment practices have long been an issue in the context of teaching and learning mathematics. It is taken almost as an axiom that what, and how, teachers assess informs learners of what is really valued and important in formal education (Cole [6]). Historical analyses, such as those carried out by Romberg [26] in the United States and Griffiths and Howson [10] in the United Kingdom, have clearly documented some roots of the assessment movement and many of its evolutionary paths. These paths mirror changing perceptions of authorities, teachers and researchers about the nature and aims of teaching and learning mathematics, the increased sophistication of the tools available to teachers to undertake assessment, and a growing awareness of how learners acquire competencies. Nevertheless, apart from excising some excesses of the past, most of the substance and philosophy of current assessment practice by teachers around the world has shown little change over time. Sometimes, purported changes are simply old ideas wrapped up with new vocabulary and technology. At other times, the changes proposed are so dramatic that teachers are not ready or willing to accept or implement them, hence they ignore the proposals and continue with their current practice. Complicating the situation further, and also impeding the implementation of new initiatives, is the linking of traditional assessment methods to the maintenance of "standards" and traditions.

Despite these impediments, there is evidence during the 1990's of an increased global focus on assessment issues in mathematics by researchers. In many ways this movement represents an evaluation of the assessment role and its many forms of practice. Reasons for this trend can only be surmised, but a plausible explanation is that the failure of many innovative curriculum and teaching reforms in the period between the 1960's and 1990's can be traced back to a lack of synchronisation between the intended curricular changes and teachers' assessment practices at that time (Niss, [20]). Hence, more information about different forms of assessment and how new approaches may be implemented, within day-to-day teaching constraints, may have many benefits.

As a possible reaction to this situation, a number of research projects in recent years have been developed throughout the world to address general and/or specific issues associated with assessment in mathematics. The

essence of this global focus has been captured by an ICMI Study, held at Calonge (Spain) in April 1991 with the theme Assessment in Mathematics Education and its Effects. At that conference, the focus on assessment was broad and a raft of issues was canvassed. The reader is directed to two publications that arose from the conference (Niss, [19] and [20]).

From this work, it emerges that the application of new assessment practices must serve the perceived needs of society. The key question here is: Who decides? There are many stakeholders, such as Governments, education authorities, employers, schools, parents, teachers, learners and researchers. It is often here that conflict occurs as different interest groups strive for recognition of a particular point of view. Hence, tensions are inevitable as important questions are posed in terms of (i) where does one find the resources for assessment (in terms of money and time), (ii) who should be the beneficiary of the assessment activity, e.g. the learner, teacher or some potential employer, (iii) how accurate are assessment practices, i.e. their validity and reliability, and (iv) how appropriate are certain assessment tasks in addressing specified goals of learning.

In this section we consider in a broad way all four of these points, referred to the context of teaching and learning geometry. We are particularly interested in assessment practices that (i) inform the learner and the teacher about the quality of geometrical knowledge acquisition and the learning experience, and (ii) offer advice on appropriate ways for instruction in this area of mathematics to proceed. At the same time, if assessment is linked to verifiable theoretical perspectives that mirror learners' cognitive growth, then issues concerning the accuracy of assessment and the appropriateness of tasks can be addressed more easily.

1. Background

As stated in the introduction to Section I, we distinguish between "assessment" and "evaluation". We adopt the more common stance that the word "assessment" relates to the process of collecting evidence on, and making judgements about, learners' needs, strengths, abilities and achievements. This definition sets it apart from the meaning of the word "evaluation", which is best described as the process of collecting evidence on, and making judgements about, the effectiveness of curricula initiatives, educational policies, and teaching programs and procedures. Assessment procedures can be either *formative*, i.e. they can occur during, and inform, the learning process, or they can be *summative*, i.e. they can occur at the end of some topic, and provide an indication of what has been learnt. Sometimes assessment has a general character, dealing with learner performance across several topics while at other times it is domain specific, dealing with a specific topic or part of mathematics. Clearly, the more topics one attempts to assess at a given time the greater the pressure to quantify the results. Hence, the information available from the assessment process to improve

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the learning environment for a given learner is less than it otherwise might be. This tension is but one further dilemma in the assessment process.

Another way to consider assessment practices, which offer a related concern to that described above, is from an ideological point of view. Here one can identify distinct paradigms which bring with them competing philosophies. At its simplest one can identify two paradigms which can be interpreted as lying at the ends of an assessment continuum. As one moves from one end to the other, features of both can be identified in varying proportions. At one end the approach is often referred to as traditional and it can be seen to dominate much of current practice all over the world, e.g. most large-population examinations and standardized achievement tests belong here. This method has a quantitative feel, is usually used to provide summative data and tends to focus upon single marks, which allow learners to be ranked. There is little advice in this form of assessment for the learner except, maybe, "work harder" if the scores are low and "keep up the good work" if the scores are high. Also, there is a reverence, by many in society, associated with such scores of an implied high degree of accuracy that does not mirror reality.

At the other end of the continuum, the paradigm is qualitative in nature. It represents a departure from an assessment focused only on learning skills, facts and algorithms to one interested in identifying and interpreting learners' understanding and reasoning. As a result, the information provided is usually in a non-numeric form that can be directly linked to the specifics of what has been taught or learnt. This means that this form of assessment is better able to inform the direction of the instruction process for both teachers and learners.

Both perspectives of assessment have their place, depending on the purposes to be addressed. In fact, although pure quantitative assessment methods dominate common practice in schools, it is not uncommon to see qualitative aspects emerging to provide support or additional information. In such cases, both understanding and memory recall are usually being assessed. For instance, diagnostic assessment (related to diagnostic teaching) is interested in knowing what deficiencies in understanding limit learners' performances within a certain mathematics domain, and what are the origins of those difficulties. This approach to assessment attempts to identify what aspects of the concepts are understood by learners, and which prerequisite skills, definitions and procedures are well understood or wrongly applied.

An important issue for the teaching and learning of geometry is its almost total reliance on the quantitative paradigm. Geometry has traditionally been considered as the part of mathematics where learners would learn formal mathematical reasoning but, in spite of this fact, teachers usually assess their learners' knowledge of geometry in ways that encourage rote

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learning. As a balance to this situation and with the purpose of encouraging the use of alternative methods, it is the qualitative perspective that underscores this section, within the context of the learning of geometry.

Our aim in this section is to address two qualitative approaches to assessment, which have been used as a tool either for formative or for summative assessment within the context of learning geometry. Specifically, our focus is on forms of assessment which allow for the interpretation of learners' responses within a framework of cognitive growth. This allows teachers to see where their learners are on some developmental ladder and, at the same time, provide advice on possible pathways for future teaching endeavours. There are possibly many models that may satisfy such a qualitative approach to the assessment of teaching and learning of geometry, but two have received attention from researchers in recent years. They have proved to be very valuable aids in assisting the development of forms of assessment of the teaching and learning of geometry that can be used by classroom teachers. They are the Van Hiele Theory and the SOLO Taxonomy. The purpose of the remainder of this section is to describe and analyze these frameworks for assessment, their similarities and differences, and their potential as assessment tools.

2. The van Hiele Theory

Since the 1980's the van Hiele theory has been the catalyst for considerable research into the learning and teaching of geometry. This theory drew upon the work of Gestalt psychology in an attempt to address the difficulties of teaching geometry. In addition, an important part of the roots of van Hiele's work is based upon Piaget's ideas and perceived problems with Piaget's theory. In particular, Piaget's *structures d'ensemble*, where learners are seen to pass through a series of stages that are closely linked to age parameters, was discounted. Instead, van Hiele attributes growth to strong social-cultural features which are linked to a teaching/learning plan that has more sympathy with Vygotsky's [28] ideas than with the biological maturation basis of Piaget (van Hiele, [27]).

In general, the main focus of research on the van Hiele theory has been directed at the notion of a hierarchy of five *levels of thinking*. The levels have been described and discussed extensively in the literature (Burger, Shaughnessy, [2]; Clements, Battista, [5]; Fuys, Geddes, Tischler, [9]; Hoffer, [13]; Jaime & Gutiérrez, [16]; Pegg, [23]). A brief summary is provided below for completeness.

Level 1: Figures are judged by their appearance and recognised by their form or shape. The mathematical properties of a figure play no explicit role in its identification.

Level 2: Figures are identified by their mathematical properties. The properties, however, are seen to be independent of one another. They are discovered and generalized from the observation of a few examples.

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Level 3: The properties of figures are no longer seen as independent. There is seen to be an ordering of the properties, with one property preceding or following from other properties. Relationships between different figures are also understood.

Level 4: The roles of the elements of an axiomatic system and of deduction are understood. Necessary and sufficient conditions can be employed. Proofs can be developed, not simply learned by rote.

Level 5: Comparison of various deductive systems can be undertaken. Different geometries can be explored based upon various systems of postulates.

In addition, the theory advocates a learning cycle of five teaching phases which encourage learner development from one level to the next.

Phase 1: *Information (Inquiry).* This part of the process allows learners to discuss th nature of the area to be investigated. It also allows teachers to be aware of their learners' previous knowledge and thinking level(s) in the topic.

Phase 2: *Directed Orientation.* Learners begin to look at the area to be studied by means of completing a number of simple tasks. The teachers' role is to direct their classes to explore the object of study.

Phase 3: *Explicitation.* As a result of the manipulation of materials and the completion of tasks set by the teacher, the need to talk about the subject matter becomes important. During the early part of the process learners use their own language. However, over time teachers assist their learners to refine their language and gradually incorporate, where appropriate, correct technical terms.

Phase 4: *Free Orientation.* Here learners are expected to find their own way to solve or undertake a variety of tasks and activities, relevant to the topic of study, which can have multiple solution paths. The teachers' role is to encourage different solutions to the problems as well as the inventiveness of learners' approaches.

Phase 5: *Integration.* The learners achieve an overview of the area of study. They are now clear about the purposes of instruction and have reached the next level. Teachers are expected to assist with the process.

While there has been little research associated with the five phases, an impressive number of studies has considered aspects related to the first four of the van Hiele levels. This work has resulted in a substantial body of evidence providing empirical evidence for the levels, although the support for other features of the theory has been more ambiguous. Some researchers, such as Burger and Shaughnessy [2], and Gutiérrez et al. [12], have even challenged what van Hiele refers to as one of the most "distinctive features" of the levels, namely, their discontinuity. As a result, a deeper knowledge of the structure of the levels has been gained.

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It would seem that van Hiele followed Piaget, maybe unknowingly, in one important aspect; namely, he sought a uni-dimensional path of cognitive development. It is as if he was trying to encompass variability within a single universal construct. From this he was able to identify and draw out features of the levels. Some of these aspects of the original characterization of his theory only became contentious when there were attempts to provide too broad a generalisation by characterising learners' understandings over a range of concepts or when the analysis of a learner's understanding was explored in too fine a detail. The implications of this being that

(i) learners may be on different van Hiele levels for different concepts, and

(ii) the van Hiele levels are not particularly suited for fine-grained analyses of a learner's understanding.

Four issues stand out that need to be considered if the van Hiele levels are used in assessment practices. First, van Hiele provided an important starting point for consideration of learner growth in geometry. By identifying a broad framework and describing a set of possible invariant structures, he has created a background from which other research can be judged. This is a necessary first step towards fully understanding learner cognitive growth.

Second, the model, as defined by van Hiele, is uni-dimensional. It offers a single universal pathway to understanding, where there are no alternative directions. Any differences in understanding can only be attributed to the rate of progress along this single path. In effect, the model proposed might be more accurately thought of as an 'ideal' or 'average' path of cognitive growth in geometry.

However, as suggested above, the results of recent research have offered a more detailed and realistic view of the van Hiele levels. In particular, it seems we are now in a position to comment upon when the transition from one level to the next is possible, and that a learner might progress simultaneously in the acquisition of understandings related to two consecutive levels (Gutiérrez et al., [12]). This new vision of the van Hiele levels is, in some ways, bi-dimensional, since learners can be assigned degrees of acquisition for a number of levels. As a necessary complement to this view of the model, new tools for the assessment of learners' acquisition of the levels have also been designed (Gutiérrez et al., [12]; Jaime & Gutiérrez, [17]; Gutiérrez & Jaime, [11]).

Third, while the assumption that all learners progress through a single path of cognitive growth is a potential disadvantage for the model, it does not suffer the same problems as other frameworks of assessment which have fixed content outcomes. Instead, for the van Hiele levels, actual specific details of the levels are not pre-determined and there is opportunity both for flexibility and diversity. The levels depend upon, and are sensitive to, the context of the learning/teaching program.

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Fourth, having assessed learners' understanding, there exists a structure in van Hiele theory upon which to base a program of instruction. This structure is provided by the teaching phases which describe a process of moving learners in an instructional setting from one level to the next.

This last aspect is significant as it offers a way forward for a teacher following the outcomes of assessment. Despite this importance, the teaching phases have not been used often, at least explicitly, to implement teaching sequences. One example of such an implementation can be seen in Jaime [15] and Jaime and Gutiérrez [18]. These authors designed a set of units for teaching plane isometries in grades 3 to 12, starting with activities of van Hiele level 1 (aimed at the recognition of each isometry as a movement and the performance of isometries with manipulatives) and ending with activities of level 4 (such as those involving deduction and proof of properties identified in activities related to previous levels and also of new complex properties, acquisition of a global view of the algebraic structure of plane isometries, and learning to do formal proofs in this context). For each van Hiele level, the activities were organized according to the phases of teaching.

For instance, when learning about rotations at level 3, a typical activity of phase 2 for this level is to ask pupils to deduce and justify that

(i) the product of two rotations is equivalent to a translation when the sum of the rotation angles is a multiple of 360°, or alternatively

(ii) the product is equivalent to another rotation whose angle is the sum of the rotation angles.

Pupils deduced the results by experimentation and observation on some examples, and they tried to validate their conclusions in an abstract way, trying to produce arguments independent of the specific examples used. Later, when pupils had learned about the relationships among rotations and products of reflections, pupils at level 3 were asked, in a typical activity of phase 4, to prove that the product of two rotations is either a rotation or a translation. Here the context is more formal, approaching the kind of formal reasoning applicable to level 4, and pupils were expected to put together several pieces of knowledge to produce an acceptable proof.

Overall, it is the integration of a consistent learning cycle in which entry conditions and exit level boundaries are determined, that sets the van Hiele theory apart from other models. It is also this feature which makes it so useful as an assessment process.

3. The SOLO Taxonomy

To interpret individual learner understandings in geometry more clearly, there appears to be a need to move beyond the broad five-level system proposed by van Hiele. This can be achieved by employing a model, consistent with van Hiele's ideas, that allows for finer-grain categorisations. Such

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a model is referred to as the SOLO (Structure of the Observed Learning Outcome) Taxonomy and was developed by John Biggs and Kevin Collis. Since the early 1980's it has undergone continual refinement and these developments have been documented in Biggs and Collis [1] and Pegg [21]. In brief, the SOLO Taxonomy is a categorisation system designed to evaluate the quality of a response to some stimulus. It has much in common with a number of post- or neo-piagetian formulations of writers such as Case [4] and Fischer and Knight [8]).

The SOLO Taxonomy has been developed independently of both the van Hiele theory and research into geometrical understanding. Nevertheless, this broader, more detailed system has been shown to be particularly effective in helping interpret learners' understanding in geometry (see, Pegg & Woolley, [25]; Pegg & Faithfull, [24]).

The SOLO Taxonomy allows learner responses to be coded in terms of two aspects. The first describes the type of thinking involved and is referred to by one of five modes, and the second describes the quality of the response within a mode by using the notion of levels. The modes are:

Sensori-motor: Associated with motor activity and can be described as tacit knowledge.

Ikonic: Associated with imaging, imagination and language development and can be described as intuitive knowledge.

Concrete symbolic: Associated with the use and manipulation of written symbol systems and is a form of declarative knowledge.

Formal: Associated with abstract constructs no longer restricted to a real world referent, the knowledge as theoretical.

Post formal: Associated with challenging and extending the theoretical constructs developed in the formal mode.

While these modes have much in common with Piaget's stages, there are at least two important differences. First, a newly developing mode does not subsume or replace earlier modes. Instead, earlier modes continue to evolve and provide support for later acquired modes. Second, these later developed modes can assist further growth in earlier developed modes.

The SOLO levels associated with each mode have the same broad description for the different modes. They are:

Pre-structural: Where the response is based on an irrelevant aspect.

Uni-structural: Where the response is based on a single relevant aspect.

Multi-structural: Where the response is based on multiple aspects which are seen as independent.

Relational: Where the response is based on relationships between several aspects.

Extended abstract: Where the response goes beyond the thinking of the mode in question and exhibits a new way of thinking.

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The specific characteristics of each SOLO level differ, depending on the mode of functioning within which the level resides. Of main interest are the three levels referred to as uni-structural, multi-structural, and relational as they lie within a mode. The remaining two levels are outside a given (or target) mode. For example, the pre-structural level refers to those responses that fail to address the question asked. As such, they may indicate thinking within an earlier developed mode or they may be an irrelevant response. The extended abstract level represents thinking of a more abstract nature and hence can usually be interpreted in terms of criteria consistent with a latter developed mode. This information is summarised diagrammatically in Fig. 1. To reduce the complexity of the diagram, both pre-structural and extended abstract levels have been omitted.

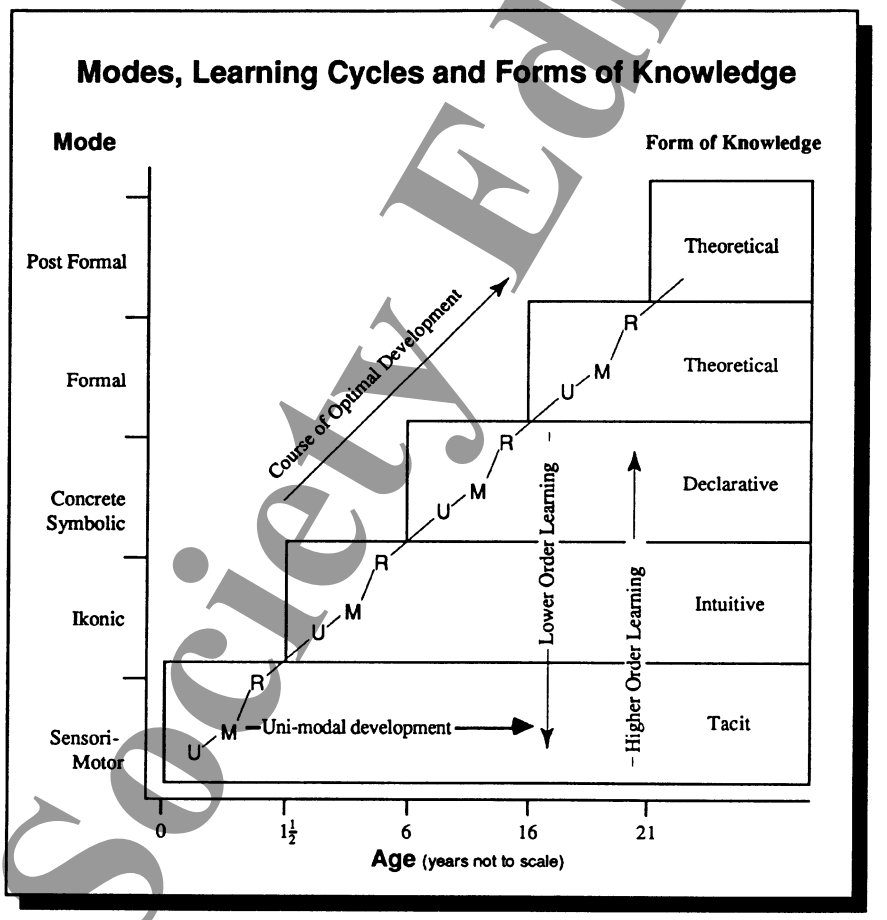


Figure 1
The SOLO model: Modes, Learning Cycles and Forms of Knowledge
(Adapted from Biggs and Collis, [1]).

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Recent work into the SOLO model has identified that there does not exist a single cycle of uni-structural, multi-structural and relational levels (as shown in Fig. 1) but that there are at least two cycles in each mode (Campbell, Watson & Collis [3]; Pegg [21]). For example, a typical growth pattern for geometry within the concrete symbolic mode can be described diagrammatically as represented in Fig. 2.

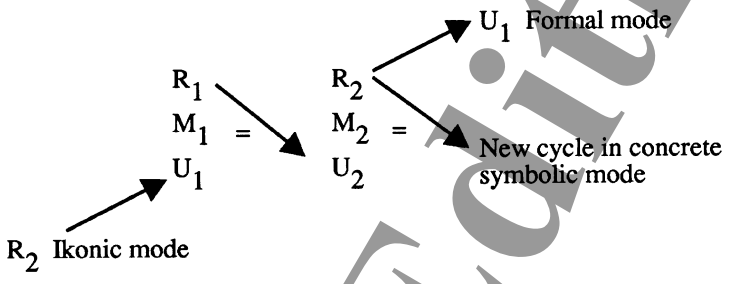


Figure 2
Diagrammatic representation
of SOLO levels associated with the concrete symbolic mode.

To explore the meaning of the levels mentioned in Figure 2, consider an example of pupils' descriptions of a certain 2D shape. A response that describes the figure by some global visual image that may incorporate features (such as, pointy, flat, corners, lines, sides, colours, and the like) would be coded at the relational level (R2) in the ikonic mode. An early response in the concrete symbolic mode involves building and elaborating upon some single feature associated with the figure. In practice this turns out usually to be sides, but it need not. "*The shape has 4 sides*" is an example of uni-structural response of the first cycle (U1) in the concrete symbolic mode. A pupil who responded consistently at this level would provide the same basic answer for a range of quadrilaterals, such as a square, rhombus, rectangle and parallelogram. The next level of response, coded as multi-structural (M1), attempts to qualify this feature (in this case sides) by attempting to bring in the lengths of the sides into the description, i.e., a pupil might consider several like shapes or compare (say) the length and breadth dimensions of the figure and then report "*This shape* (a rectangle) *has the top and bottom sides longer than the side sides*". Responses are coded as relational (R1) when a single property is mentioned, e.g. "*A square has all the sides around it the same length*". This first cycle of development in the concrete symbolic mode has many links to the ikonic mode and represents a transition from an intuitive feeling of space to the development of geometric thinking. In addition, there are usually references to images associated

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with the above responses, especially if the pupils are pushed to provide additional information. This cycle also charts the growing emergence of the mathematical concept of 'property' as it pertains to shape identification.

The second cycle commences with a response that succinctly states a single property, e.g. "*A square has four equal sides*". This response is coded as U2 but often it is difficult to distinguish between responses at R1 and U2 without an interview. The essence of the difference hinges on the ability of a U2 response to be less drawn out (less verbose) or convoluted and more precise than an R1 response.

The example above illustrates coding in the first cycle based upon the development of the feature, sides, beginning with identification of the number and ending with the incorporation of length considerations. At the same time, pupils are developing property concepts related to other features. These are associated with features such as corners or the gap between lines. This development results eventually in properties related to angles and parallelism, although these are usually not mentioned in relation to shapes, at this level, unless pupils are specifically prompted to do so. The next level of response, coded as M2, includes the spontaneous reference to several properties, i.e. the various properties developed individually during the first cycle become overt. This marks the ability of a pupil to link, or list in a sequential fashion, several properties. There is no indication here of a relationship between these properties. Finally, relationships between properties resulting in descriptions that indicate an overview (coded as R2) are reported.

The difference between the two cycles identified in the concrete symbolic mode can best be articulated by comparing the response levels R1 (from the first cycle) and R2 (from the second cycle). Both levels indicate a combining together and overview of aspects within the relevant cycle. R1 represents an end product resulting from combining various aspects which culminate in pupils being able to provide a single property in a non-prompted situation. An R2 response represents an end product of combining several properties so that relationships between these properties as well as between figures are made explicit.

In Figure 2 there are two paths indicated for responses whose quality is an improvement on an R2 response. One path is to provide a U3 response which would be an initial response in an hypothesised third cycle in the concrete symbolic mode. A second path would be to provide a U1 level response in the formal mode. In this latter case one might expect to see the beginning emergence of the use of necessary and sufficient conditions.

The detail provided by the SOLO structure offers flexibility and versatility in categorising responses. In particular, four issues stand out which are relevant to assessment. First, in SOLO categorisations, the target of assessment is the quality of the structure of learners' responses. This notion avoids the problem of *décalage* associated with Piaget's ideas (and van

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Hiele's), as a respondent is not expected to respond consistently at a given level across a number of topic areas or within the same topic on different occasions. Instead, responses are sensitive to the motivation of the learner, the amount of experience with the material, the learners' interpretation of the stimulus item, and so on.

Second, the categorisations possible within the current formulation of the SOLO taxonomy are extensive. Not only are there at least six levels within the concrete symbolic mode but there are similar numbers of levels in the other modes.

Third, the taxonomy has multiple dimensions. Not only is there the opportunity to describe the quality of responses within a mode but the interaction of other modes allows for an explanation of diversity and individuality in learners' responses that has not previously been available.

Fourth, while there is no teaching scheme associated with the SOLO model, the additional arrows in Fig. 1 imply important general principles about learning. For example, with regard to the diagram, there is potential for bottom-up and top-down support to improve learning in some target mode for a particular skill. In a teaching situation the majority of learning experiences should be directed at the target mode. Nevertheless, it is valuable to ensure that relevant skills in a previously acquired mode are available before learning has commenced and that it provides ongoing support, where appropriate, to enhance learning. Alternatively, it may enrich the learning experience if more abstract skills, which have been previously acquired by the learner, are used to enrich the development of new skills whose target area is associated with an earlier mode.

4. Towards an integrated assessment in geometry

From the previous descriptions, links can be established between the van Hiele Theory and the SOLO Taxonomy. For example, the van Hiele levels can be matched to SOLO modes (Pegg, [22]):

van Hiele levels	**SOLO modes**
Level 1	Ikonic (I)
Level 2 and 3	Concrete symbolic (CS)
Level 4	Formal (F)
Level 5	Post Formal (PF)

It is possible also to make a more elaborated matching by allocating SOLO levels to the responses that indicate van Hiele levels. The state of the current research means that this is easier with van Hiele levels 2, 3 and 4; however, there are indications that would allow us to hypothesise about the SOLO connections with van Hiele level 1.

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The different degrees of acquisition of level 1, where learners recognise figures by their global appearance, can be interpreted within the levels U2, M2 and R2 in the ikonic mode. It would seem that these three SOLO levels may represent a focus on some single aspect of a shape, several unrelated aspects and an overview, respectively. The word "aspect" is used to describe words such as pointy, flat, corners and sharp and these are very different from mathematical properties that are a feature of the concrete symbolic mode. The SOLO levels in the first cycle of the ikonic mode, namely, U1, M1 and R1, can be possibly interpreted as a van Hiele level 0, defined by some investigators (Clements and Battista, [5]) to describe learners who still have not started the acquisition of level 1, i.e. they show levels of response which indicate that the imagery process is quite incomplete.

Level 2 in the van Hiele theory, where learners recognise figures by their properties, can be coded at either U2 (CS) or M2 (CS). In the case of a U2 (CS) response, i.e., when learners have a low acquisition of level 2, they base their observations on a single property. When the response is coded at M2 (CS), i.e., when learners have a medium to high (or complete) acquisition of level 2, observations can be made on the basis of many properties. The earlier U1, M1, R1 (CS) cycle of SOLO is a development that is in transition between seeing a shape as a global picture (van Hiele level 1) and working with a single mathematical property (van Hiele level 2).

Level 3, where learners see relationships between properties or figures and can undertake informal proofs, can be coded at R2 (CS). Learners with complete acquisition of level 3 are able to provide simple formal proofs; however, the main differences between this and level 4 is that the learners at level 3 can only use relationships, such as congruency, in relatively straightforward prompted situations and where there is an empirical referent.

The process of acquisition of level 4, where learners understand the deductive process and can use relationships as a tool in the solution process, can be coded by at least two cycles U1 - M1 - R1 and U2 - M2 - R2 in the formal mode (F). The first cycle in this mode represents a development of the deductive process, where a R1 (F) response describes a learner who can competently apply relationships, such as congruency and similarity, in addition to being able to keep track of most conditions inherent in the question. A U2 (F) response appears to relate closely to the successful application of necessary and sufficient conditions in non-prompted situations, and is possibly the best response that capable learners can achieve in the secondary school.

Level 5, where learners challenge old, and adopt new, axiomatic systems is hypothesised to be represented by cycles of SOLO levels in the post-formal mode (PF). Supposedly, level 5 could only be achieved by professional mathematicians or bright learners, although experiences of persons reasoning at this van Hiele level have not been reported.

Secondary Edition

Traditionally, researchers have devoted considerable effort in preparing reliable tools to assess learners from the point of view of either van Hiele levels or SOLO categories. Such tests (paper and pencil, or interview) have been designed to assess learners in different areas of geometry. While an important aim of such research is to close the gap between research and practice, teachers often are unable to use such methodologies due to various constraints, the short time for assessment being one of the most important. One attempt to give an answer to teachers of geometry, in how they might assess their learners' reasoning and knowledge, has been carried out by designing a single test that can be depicted from the points of view of both the van Hiele model and the SOLO taxonomy (Huerta, [14]). This test was administered to a sample of 74 pupils and students from Primary to the University (Table 1).

Education level	Average age	Number
Primary	11 years	20
Secondary	16 years	19
Secondary	18 years	23
University	21 years	12
TOTAL		74

Table 1

The instrument was designed to address both the requirements for van Hiele tests (Jaime & Gutiérrez, [17]) and the organization of SOLO tests (Collis, Romberg & Jurdak, [7]). It consists of five super-items, with four questions in each super-item. Two super-items deal with parallelism and three super-items deal with quadrilaterals. Each super-item assess the grade of acquisition of van Hiele levels 1, 2, and 3, and some questions assess also level 4. The questions in each super-item are organized so that the reasoning requirements, and structure of response, necessary for correct answers, increase in complexity as learners move through the super-item. From the point of view of van Hiele, the first questions require reasoning in lower levels than the last questions. Fig. 3 shows a typical super-item used in this test. From the SOLO point of view, the first question assesses answers at the uni-structural level, since learners have only to use one datum from the stem. The second question assesses answers at the multi-structural level, because learners have to use two properties, although they need not to be linked one to the other. The third question assesses answers at the relational level, because angle C may only be calculated after relating at least two angles and at least a property from the stem. The fourth question assesses answers at the extended abstract level, since it asks the learners to prove a property of triangles.

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STEM

When a pair of parallel lines are cut by a transversal (Figure 1), we have eight angles verifying the following properties:

- The alternate interior angles are equal: B = F, G = C.
- The alternate exterior angles are equal: A = E, D = H.
- The opposite angles are equal: A = G, B = H, C = E, F = D.
- The corresponding angles are equal: F = H, C = A, E = G , D = B.

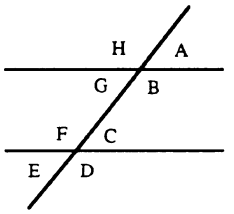


Figure 1

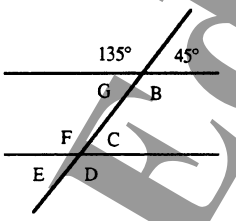


Figure 2

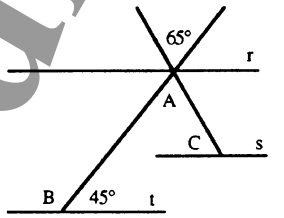


Figure 3

QUESTIONS

1. Determine the value of angle G in Figure 2.
2. Determine the value of the angles D and C in Figure 2.
3. In Figure 3, the lines r and s are parallel and the lines s and t are parallel. Determine angles A, B and C.
4. Prove the following property for every triangle: "The sum of the three angles of a triangle is 180°".

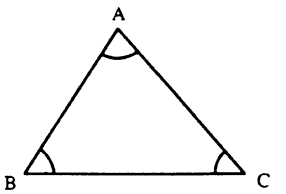


Figure 3

The learners could answer question 1 by reasoning at van Hiele levels 1 or 2. Some answers are:

G = 45° because G is beside A and so G measures 45° like A (Level 1; R-I).

G = 45° because they are opposite angles (Level 2; R-CS).

G = 45° because in opposite angles it is said that A = G. So, if A = 45° then G = 45° and both have the same value (Level 2; M-CS).

G = 45° because A = G and A = 45°, so G = 45° (Level 2; U-CS).

The first answer shows a possible reasoning in van Hiele level 1. The learner does not use a property to address the question, but only uses physical aspects of the figures in the answer: The value of G is justified by the position of the angles. From the point of view of SOLO, this answer is relational (R) in the ikonic mode.

The other answers show reasoning at van Hiele level 2. At the same time, they show an internal structure in the concrete symbolic (CS) mode. The second answer shows a global understanding of the property "when two lines intersect, opposite angles are equal" since the learner mentions such a property to justify that G = 45°. So, this answer is relational (R1). In the third answer, the learner explains the property by using the appropriate equality among those exemplified in the stem. He uses the information as separated aspects, so this answer is multi-structural (M1). Finally, in the fourth answer, the learner only mentions the equalities to answer, showing that he is only able to use an aspect of the information. So, this answer is uni-structural (U1). Finally, learners who did not solve the question were considered as pre-structural (PRE).

The assessment from the van Hiele point of view gave a distribution of the learners in several profiles of reasoning. Table 2 summarizes the results of the assessment from van Hiele perspective. The above mentioned profiles are:

Profile 1: Learners with a complete grade of acquisition of van Hiele levels 1 and 2, and indications of reasoning in other levels.

Profile 2: Learners with a complete grade of acquisition of van Hiele level 1, and indications of reasoning in other levels.

Profile 3: Learners with no complete acquisition of van Hiele level 1, and indications of reasoning in other levels.

Profile	Number	%
Profile 1	18	24.3%
Profile 2	33	44.6%
Profile 3	23	31.1%
TOTAL	74	100%

Table 2

Distribution of the learners' van Hiele levels of reasoning.

After assessing the learners with the SOLO Taxonomy, two learning cycles in each concrete-symbolic and formal modes were found in learners' ways of use of geometrical properties. Table 3 summarizes the results of the assessment with the SOLO Taxonomy for the learners in each van Hiele profile. Also a parallelism was found to exist among the van Hiele profiles and the SOLO levels.

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		Ikn.	Concrete-Symbolic						Formal					
	PRE	R	U_1	M_1	R_1	U_2	M_2	R_2	U_1	M_1	R_1	U_2	M_2	R_2
Pr. 1							1.3		4.1	1.3	2.7	1.3	9.5	4.1
Pr. 2		10.8		1.3		10.8	9.5	4.1	5.4		2.7			
Pr. 3	6.8	5.4	1.3	1.3	1.3	4.1	1.3	6.8			2.7			
TOT	6.8	16.2	1.3	2.6	1.3	14.9	12.1	10.9	9.5	1.3	8.1	1.3	9.5	4.1

Table 3
Distribution (%) of the learners' van Hiele levels
of reasoning and SOLO levels of answer.

5. A Way Forward

We are entering a period when new approaches, incorporating strong philosophical positions, are emerging in the teaching and learning of mathematics. In many of these, learning mathematics is seen as much more than an acquisition of sequential succession of elements of knowledge. Such frameworks encourage the teacher to consider multiple ways for their learners to acquire knowledge. They also place an emphasis in teaching on learners being able to demonstrate that they understand the mathematics they meet by their ability to use and apply their mathematical knowledge. This is an aim that is very consistent with the views of van Hiele and one he would refer to as the development of insight in mathematics.

For such reforms to be successful there needs to be a close affinity between learners' potential abilities, the teaching intentions of the mathematics curriculum, its philosophy, and the range and type of assessment practices used. If teaching is to promote many ways of learning, the associated assessment should also take many forms. The models of assessment derived from SOLO and van Hiele theories, as described above, offer a chance to provide alternative forms of assessment which are a balance to more traditional methods.

Many of the features of the two proposed models of assessment are self evident but some deserve mention.

1. Assessment should not only be an end in itself. Instead, it should also inform the instruction process. This formative role is far more sympathetic to emerging views of what it means to know and learn mathematics. An important consequence of this is that assessment procedures should contain qualitative elements. Both van Hiele and SOLO models offer teachers the opportunity to address this issue, since they integrate assessment criteria and recommendations on the organization of teaching.

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2. The focus of assessment in both models is on the structural level of learners' learning, i.e., attention is on how the knowledge gained by learners is organised in coming to a decision or solution. It is not about labelling people into boxes from which it is difficult to escape.

3. Both models allow the teacher (and possibly the learner) to identify the quality of responses and hence describe where a learner is currently performing. This focus is positive, being in terms of what a learner knows and how the learner uses such knowledge.

4. The consequences of being able to ascertain a learner's understanding in terms of a picture of overall cognitive growth is a valuable feature of both models. It also means that the teacher is better informed about where the next stage of instruction might most profitably commence and where it should head.

6. Conclusion

Both the van Hiele theory and the SOLO taxonomy are means of categorising a learner's achievement. They build upon the current empirical research base into learners' understanding. The identification of discrete levels in both models suggests that there are a relatively small number of identifiable categories when analysing learners' responses to a given question and, as such, these groupings provide valuable information to teachers and learners. When this information can be interpreted within established frameworks, the potential benefits are increased and the validity of the process enhanced.

Quite often frameworks, such as those offered by the two models, are seen as belonging only to the domain of researchers. They are seen as appeasing a lust for theory in academics but as largely irrelevant to teachers, who see themselves battling numerous day-to-day problems. Discussion about philosophies and theories is often seen as esoteric and out of touch with what "is really going on in schools". Even those ideas that are seen as having relevant practical value can still be very difficult for teachers to implement. This can arise because:

(i) the method of assessment does not fit the specific requirements needed by the teacher for reporting;

(ii) it may be very time consuming;

(iii) it may require an understanding of the topic or the nature of learners' responses that is not accessible to the teacher;

(iv) the teacher may need to report on a broad range of issues; and,

(v) the techniques used to assess learners may not be appropriate for a school situation.

There is no attempt in this section to devalue these concerns or to minimise difficulties inherent in the use of qualitative models of assessment. Just as assessment practices should match teaching intentions, so should

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researchers try to accommodate practical issues that confront teachers. We have tried here to move a step forward in the direction of linking research findings to practice, by offering a way to integrate two models of assessment in geometry which have a strong qualitative basis. Further, we have suggested that both these models can play a vital role in providing support for teaching techniques which can enhance learner learning if assessment policies reflect this philosophy as a primary purpose for assessment.

It appears that the mathematics education community has reached a critical watershed in terms of the direction mathematics instruction and assessment should take. Questions that require answers include: What should be the objectives of teaching geometry in primary and secondary grades? Should curricula move from the traditional view of geometry as the field where logic and formalism have to be learned? How can we improve how we teach and assess? Are we experiencing a shift in focus in the teaching of mathematics away from a predominantly quantitative approach that values the application of routine algorithms and the completion of a number of similar exercises using identical methods in a lesson to one that values variety in questions and encourages multiple solution paths? If this shift is happening in teaching, should assessment practices follow the same trend? An approach to teaching geometry which is consistent with the van Hiele and SOLO models of assessment seems to offer the potential for learners to have ownership of the mathematics they encounter, to show insight, to use higher-order skills (as opposed to just rote-learning), and to use their geometry in creative ways.

The task ahead is not easy and requires close cooperation among many of the stakeholders, including practising teachers and researchers. However, the goal, namely, more learners knowing mathematics (in particular, geometry) better, seems very much worth the effort. The van Hiele Theory and the SOLO Taxonomy, either taken together or separately, seem an appropriate catalyst for such work. Clearly, much more research needs to be done with teachers to fully develop the potential of both models for interpreting, assessing, and improving the teaching and learning of school geometry. Nevertheless, even as they currently stand, these models offer considerable potential to improve both teaching and learning of geometry, and, in particular, as a qualitative tool for improving the assessment of learners' understanding in geometry.

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9. TEACHER QUALIFICATIONS AND THE EDUCATION OF TEACHERS

Mogens Niss

1. Introduction

The task of the present chapter is to consider teacher qualifications and the education of mathematics teachers with particular respect to *geometry*. This is not an easy task to accomplish, since many of the important factors and issues to be considered are associated with the teaching of mathematics in general – geometry being just a special case – or even with the profession of teacher as such. Thus, it would not make sense to insist, for this chapter, on restricting the analysis exclusively to matters specific to geometry. Firstly, this would require us to establish a clear demarcation line, not easily drawn at all, between "geometry as such" and "geometrical aspects" of other mathematical topics or branches. Secondly, although the focus of this book is on the teaching of geometry, in most places in the world geometry is, and probably will continue to be, taught as part of mathematics by teachers who teach other branches of mathematics as well. Hence their qualifications as teachers of geometry are imbedded in their qualifications as teachers of mathematics. Nevertheless, in what follows we shall attempt to concentrate our attention on geometry, as much as is possible and reasonable for the avoiding of a distorted treatment of our theme.

As is discussed elsewhere in this book, the place of geometry in mathematics curricula, at all levels, has changed dramatically more than once during the last four decades. Corresponding changes have taken place in the preparation and qualifications of mathematics teachers for primary, secondary, and tertiary levels, albeit usually with a time lag. Before, say, 1960 geometry occupied a prominent (in some places even a dominant) position in all categories of post-elementary mathematics education in most countries. Similarly, the preparation of mathematics teachers to teach geometry formed a substantial part of their pre-service teacher training. Then, in those countries in which the New Math movement found a foot-hold in the 1960's and 1970's, the situation changed. Soon, in those places, geometry became either 'integrated away' into more general and abstract theoretical edifices, such as linear algebra or transformations groups, in which the service paid to geometry was often lip-service mainly, or it simply disappeared

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from curricula. This was reflected in parallel changes in teacher-training programmes, agendas of in-service courses and conferences etc. It was further reflected in university programmes. In many tertiary institutions geometry disappeared altogether, or it survived under headings such as "differential geometry", "algebraic geometry", or "convexity". Such topics were often taught, however, without any reference to, let alone a basis in, the systematic study of specific geometrical objects or 'worlds'. In the same vein, research mathematicians only seldom called themselves "geometers". Rather, depending on their specialty, they preferred to see themselves as "topologists", "global analysts", or "algebraists". If geometry disappears from university mathematics, it is also likely to disappear from the preparation of those who are to teach mathematics at earlier levels of the education system. So, the situation in university departments tended to reinforce the evaporation of geometry from mathematics education at large.

It is true that this picture was not uniform across countries. Some countries, (e.g. China, Hungary) never completely severed the umbilical chord connecting their curricula to the 'pre-modern' tradition with its substantial geometrical component. So, in these countries, during the last decade, the gradual revival of the attention paid to geometry by mathematics educators does not represent the same degree of discontinuity as is the case with countries that embarked on the 'modern mathematics' reform. The revival of geometry which began in the mid-1980's is still of a fairly modest magnitude in most countries, although the movement is gaining more momentum. The term 'revival' should not be taken to imply the unchanged resumption of past traditions of the teaching of geometry. What is revived is the bringing back of geometrical objects, phenomena, problems, theories and methods as subjects of study in mathematics education. (The manifestations of, and reasons for, this development are discussed elsewhere in this book, but undoubtedly one key factor resides in the possibilities of visualisation offered by information technology and the ensuing opportunities and challenges for the teaching and learning of mathematics. From that perspective, what we are experiencing is not just a revival of geometry but a revitalisation as well.)

So far, the changes in the preparation of teachers seem to be scattered. No systematic innovation appears to be on the agendas of teacher trainers. However, things are moving. Thus many universities (e.g. Cornell University, USA, as reported by David Henderson[1]) have established a variety of new courses in geometry in addition to the 'modern traditional' ones in differential or algebraic geometry. The didactics of geometry has become included in the programme of many institutions responsible for the training of mathematics teachers. But again, we are hardly witnessing a universal reform movement in the making (the experience with the New Math reform has probably resulted in a long lasting distrust in large-scale,

[1] This information was given in a plenary presentation at the ICMI Study Conference *Perspectives on the Teaching of Geometry for the 21st Century*, Catania, Sicily (Italy), 28 September - 2 October 1995.

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top-down curriculum reforms). Rather, there is a growing number of innovation initiatives which are beginning to form a pattern.

2. What are the issues?

In order to clarify the issues we shall deal with in this chapter, it might be worthwhile to say a few words about what we are not going to do. First of all, it is important to *avoid the 'syllabusitis trap'* [(2)], i.e. the trap of discussing matters of mathematics education (in this case with respect to teachers) solely in terms of syllabi and curricula. Identifying mathematical knowledge, insight, and ability with a list of the topics, concepts, results, and methods that people ought to master, implies an unduly simplistic reduction of what mathematical competence is all about. This is quite similar to the reduction implied by the identification of linguistic competence with a vocabulary to be known and a list of grammatical rules to be mastered. So, as far as geometry is concerned, to describe the qualifications and the education of a mathematics teacher should not be equated with listing the geometrical topics, theorems, facts, and methods which the teacher has to know or, at least, has to have been exposed to. In essence, this is a question of not confusing necessary and sufficient conditions. Of course, in the same way as it is impossible to possess linguistic competence in a certain language without knowing some basic words and some basic grammar, it is crucial to know some geometry in order to teach geometry. However, as our ambitions, in this context, go beyond the mere indication of elementary necessities, we have to search deeper in order to characterise *what it takes* to be a competent mathematics teacher of geometry, as well as what it takes to educate such a person.

Here, we have to beware of another trap. It is no great art for mathematics educators to formulate and to agree on a huge gross list of desirable qualifications concerning geometry and its teaching and learning, with which the mathematics teacher of the 21st century should be equipped. Similar lists of qualifications regarding the teaching and learning of algebra, calculus, probability and statistics, applications and modelling, or the history and philosophy of mathematics, etc. could be established equally easily. It is probably true that the combined list of such qualifications would characterise a marvel of a mathematics teacher whom it would take several handfuls of years to educate – and a good deal of motivation and improved working and living conditions to attract – to the profession of school teacher. In other words, it is not too difficult to take off to Utopia. The difficult part is to get back to reality.

None the less, even if Utopia is hardly an appropriate habitat for permanent residence it might well be worth a visit once in a while. Occasional visits to Utopia serve one important purpose, among others (cf. Niss, [3]):

(2) The term 'syllabusitis' was coined in Lewis [2]. The author is grateful to his friend and colleague Jens Hojgaard Jensen for having pointed out the term and the reference.

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When, in a real world in which not all wishes can be fulfilled at the same time, we are to design, organise and implement ways to educate mathematics teachers, it is important to be able to assign priorities to all the various desirable qualifications we can suggest, in order to identify the more crucial ones and to know what we lose with the ones we have to give up. If all decisions concerning teacher education were to take only the conditions and circumstances of the day into account, our level of ambition would tend, to an increasing extent, to be lower than should and need be the case. Ultimately, the education of teachers would be governed by a coalition of the law of inertia and political-economic power plays.

Against this background, the main issue of this chapter can be phrased as follows:

How to ensure that future teachers of mathematics will have a multi-faceted and well-founded view of geometry, an ability to stage a multitude of rich teaching-learning environments in relation to geometry, and an ability to assess pupils' or students' geometrical understanding, knowledge and ability?

For this question to make sense, we have to clarify what the key items "future teachers of mathematics", "a multi-faceted and well-founded view of geometry", "rich teaching-learning environments in relation to geometry" and "an ability to assess pupils' or students' geometrical understanding, knowledge and ability" mean, or could mean. We shall deal with this task in the sections below.

Firstly, the term "future teachers" is not well-defined. Here, it should be taken to encompass different categories of people: The teachers who are, already, in service today but will be serving in the future as well; the pre-service student teachers of today who are to function tomorrow and in a more distant future; and the yet-to-come prospective teachers who will teach future generations of pupils and students. This implies that we are not only talking about the qualifications and the pre-service teacher-training of today and tomorrow, but also about the qualifications and the in-service education of teachers today and in the future.

Although the teacher education issues we are dealing with are certainly pertinent to all sorts of teachers, from kindergarten to university, for simplicity our main focus will be on the qualifications and the education of school teachers (grades K-12). From time to time, however, we shall also address the education of tertiary level teachers, including teacher trainers and university professors of mathematics.

Our attempts to clarify what "a multi-faceted and well-founded view of geometry" means will be carried out as part of §4: *Knowledge and views of geometry as a mathematical discipline.* Similarly, "rich teaching-learning environments in relation to geometry" will be dealt with in §5: *Knowledge and views of processes of learning geometry* and §6: *Rich teaching-learning*

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environments in and with geometry. A separate §7: *Teacher assessment in geometry* will offer a discussion of "the ability to assess pupils' and students' geometrical understanding, knowledge and ability". Perhaps the final paragraph of the Chapter is the key one: §8: *Ways to develop a competence to teach geometry.*

3. Some background assumptions

Normally, a teacher will *function for several decades*. During his service, *substantial changes are likely to take place* in the political, socio-economic, technological and cultural conditions in society, in the education system, in mathematics as a pure and applied science, in the mathematics curriculum, and in our knowledge and understanding of the teaching and learning of mathematics. Probably even the very place, role, and task of a teacher will be subject to fundamental change, the nature of which is, necessarily, obscure to us today. Nevertheless, we have to form and discuss visions of the future of the teacher in general and of the teacher of mathematics in particular.

At any rate, even if changes were less drastic than might turn out to be the case, no teacher can expect his pre-service education, however thorough and well-founded, to provide *sufficient preparation* for his entire career as a teacher of mathematics (cf. Niss, [3]).

Sometimes, in some countries, society wants its teachers to be obedient instruments for the realisation of society's specific ideology, purposes and interests concerning (mathematics) education as coined in curricula, syllabi, assessment modes and so forth. In such cases, teachers are not wanted to be independent, or to have a critical and reflective attitude towards the conditions, framework, organisation and content of mathematical education. They are not expected to take initiatives on their own beyond the constraints laid down in the regulations established from above/from outside, let alone to devise curriculum segments, teaching-learning activities, or teaching materials, themselves. Succinctly put, not every society wants its teachers to have self-confidence, or to exercise it in case they have it.

In contrast, it is one of the axioms of this chapter that *teachers should be educated so as to be self-confident*, independent – yet certainly not resistant to new knowledge or reasoning – energetic, and enthusiastic *professionals in the practice of mathematics education.* Put concisely, teachers should be educated for a profession rather than for a job, and to become intellectually autonomous rather than echoes of current trends and fads. This axiom has its justification in centuries of empirical observation. Teaching of mathematics, and hence of geometry, given by teachers who are not self-confident will inevitably degenerate into stereotyped rituals which cannot be adapted or revised in response to changing conditions, circumstances or challenges, neither on a small nor on a large scale. In other words, self-confidence is

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a prerequisite to flexibility and multi-perspectivity in teaching. It also requires self-confidence to answer "I don't know but I can probably find out" to pupils' or students' challenging questions, and to take the risks involved in engaging in open-minded excursions to land not previously visited by the teacher. And perhaps flexibility and open-mindedness is more needed for the teaching of geometry than for most other mathematical subjects.

4. Knowledge and views of geometry as a mathematical discipline

The nature and role of geometry as a discipline endow it with a unique position in mathematics. As mentioned in many chapters of this book, geometry is several different things at the same time. It is not just a mathematical topic. Geometry is a mathematical theory (or rather several theories) of *physical space*, at micro (the atomic world), meso (the human world) and macro (cosmological world) level. Geometry is a collection of features and properties of *physical objects*, whether they already exist in nature or are man-made constructs. In these respects, geometry forms part of *natural science* as well as of the engineering and *design sciences* in a broad sense. In both cases, geometry is an infinitely rich source for *modelling* of natural and artificial reality, ranging from ancient patterns in pottery or textiles to the intrinsic structure of the universe. Geometry is a special *separate branch* of mathematics, consisting of different interrelated theories with different sorts of foundation. These theories include axiomatic or synthetic Euclidean and non-Euclidean geometry; analytic geometry and vector spaces; projective geometry; algebraic geometry; differential geometry; finite geometries; combinatorial geometry; and so forth, in addition to various unifying 'umbrella theories' of, so to speak, the second order. Geometry is further an aspect of, or a perspective on, almost every other branch of mathematics. This is reflected, for instance, in the very fact that so many mathematical objects are labelled with the term "space" (e.g. "function space" or "measure space"), and in the fact that concepts such as "dimension", "orthogonal", "volume", "projection " etc. are used in contexts which bear no direct relationship to geometry in its classical meaning. Finally, geometry is a constitutive component in *human vision and visualisation*, and hence in all attempts to represent, describe and understand these, for example in perspective drawings and computer graphics.

Within mathematics, hardly any other branch occupies a similar omnipresent, complex, and significant position as does geometry, (with the exception of numbers, arithmetic and algebra). This gives rise to special requirements and challenges to teachers' knowledge and views of geometry. Regardless of the educational level at which the teacher works, or is going to work, it seems crucial to us that he possesses some form of experience, knowledge, and views of the fact that the dimensions just referred to are

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present as characteristic but different facets of geometry as a topic. It goes without saying that this does not imply that those who teach geometry, in the entire range from kindergarten to Ph.D studies, all ought to have the same degree of in-depth and detailed experience, knowledge, and views of geometry as a natural science, a design science, a multitude of theoretical structures each with its own specific basis and body, an all-pervading aspect of mathematics at large, and a fundamental component in human vision and visualisation. Of course not. In addition to being a pretty unrealistic requirement, it would also be an unreasonable exaggeration of the needs of, say, a primary school teacher. However, what is implied is that the dimensions as such and the differences between them should form part of the education of, in essence, any mathematics teacher irrespective of the level at which he teaches or will be teaching. Furthermore, at any level of education, geometrical competence takes different forms, ranging from "acquaintance", over "being knowledgeable", to "having mastery" (this goes for teachers as well as for pupils and students). The important thing is that each of these dimensions and forms of competence can be materialised in numerous ways corresponding to the teaching level at issue. Naturally, the higher the level of teaching, the more comprehensive, multi-faceted, and complex is the image of geometry that is on the agenda of teaching and learning. Accordingly, the requirements on teachers' geometrical insights increase in extent and depth with the levels on which they teach.

The teaching of geometry at primary school, in most countries, emphasises the exploration, naming, description, classification, drawing, mensuration (sometimes accompanied by simple computations) of concrete physical objects in the plane or in the space. Thus, it is the naturalistic and constructive facets of geometry that are in focus of the attention of pupils and teachers. Primary school teachers, therefore, will tend to see geometry as mainly a natural science and as a design science, by and large of an empirical nature. Generally speaking, it is certainly reasonable to concentrate on these facets in primary school geometry. However, primary school teachers should be educated so as to be aware that geometry is not merely a collection of empirical facts which form an experiental discipline. They need to know that essential parts of the geometry of physical objects in physical space can be subjected to deductive reasoning and be organised as an extensive theoretical structure in various ways. They need to know how geometrical considerations may shed light of immediate insight ('aha! experiences') on arithmetical operations, rules, phenomena, and results. They need to know that human vision is, in crucial respects, governed by geometrical laws and properties which account for the striking similarity between our visual images and photographs, pictures provided by electronic media, perspective drawings and paintings, and so forth. Already at the primary level, the relationships between 3-dimensional objects and 2-dimensional

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images are of key importance to the fostering of basic geometrical knowledge and understanding.

The point here is not that teachers have to know all this in an elaborate technical sense, let alone that they should necessarily teach it to their pupils. Rather, these perspectives should be considered as a surplus of knowledge and background which will help teachers to create richer and more stimulating learning environments with respect to the things they do teach. The more facets and connections their own geometrical backgrounds contain, the more will they be able to encourage and guide their pupils to experience, explore and examine geometrical matters; and the more will they be able to sow seeds in their pupils which, in later stages of schooling, can grow to widely ramified plants of geometrical knowledge and insight.

When it comes to teachers on higher levels of the education system, the spectrum of dimensions they need to know about in specific terms obviously becomes broader. At the same time, the depth of their geometrical experience, knowledge, and views will have to be increased with the level. Furthermore, they will need to possess larger sets of examples and cases drawn from a wide variety of contexts and frameworks pertinent to geometry. At higher levels, they need to know something about different geometrical theories and their inter-relations. Especially, the transfer between different kinds of geometrical "codes" and "modes" – such as "visual observation, intuition and representation", "physical geometrical modelling", "analytic or vector geometry", "deductive Euclidean geometry", "paper folding geometry", or "transformation geometry" – becomes important at the post-elementary levels, and hence in the preparation of teachers. Even if these expectations may well appear to be self-evident in theory, they are less so in practice. Apart from the fact, discussed in the introduction, that geometry still holds a fairly humble position in many teacher-training programmes throughout the world, it often so happens, in the cases where geometry is present, that only its theoretical dimensions are represented in the programme. Thus, at higher levels, it is not unusual that geometry is split into a (not very large) number of branches, each taught simply as yet another type of theoretical mathematics.

So, if the problem with the geometrical education of primary school teachers lies mainly in the absence of knowledge of geometrical theory, the problem tends to be the reverse with teachers on secondary and tertiary levels. If they have any geometrical education it is likely to be predominantly theoretical, whereas the other dimensions of geometry, equally important in the context of education, are usually greatly under-represented both in pre-service training and in in-service programmes. Of course, it is important that teachers have a solid and varied background in different species of theoretical geometry. For instance, any post-primary mathematics teacher should be acquainted with instances of axiomatic as well as of analytic geometry, and of the fact that not all geometry is Euclidean. Moreover, he

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should be able to subject pure or applied mathematical problems to different kinds of geometrical treatment as appropriate or convenient in the situation. But, in order to be able to endow their teaching of geometry with richness in quality and perspective, and to exercise it with autonomy and self-confidence, (cf. §3), their knowledge and views of the discipline need to encompass also other dimensions of geometry than the purely theoretical ones.

In summary, we want to advocate that mathematics teachers on all levels be educated in such a way that they acquire a representative, multidimensional and balanced knowledge and view of geometry as a mathematical topic in all its dimensions as outlined above, with due attention being paid to the level at issue. Teachers should neither perceive geometry as only an empirical science of nature or design, nor merely as a separate theoretical mathematical discipline like so many others. Finally, teachers should be aware that geometry has a long, winding and remarkable history, as a discipline which has its origins in the earliest human communities and civilisations, and which ever since has been deeply imbedded in human culture, society, science and technology. Generally speaking, in geometry, as in any branch of mathematics, teachers need to know and understand a good deal more than their pupils or students. They need a reserve of knowledge, insight, ability, and experience, which goes beyond the immediate day-to-day requirements.

In this section we have emphasised the principal dimensions of geometry which we believe that teachers should have either insight in, knowledge or view of, or a feel for, depending on the circumstances, in order to acquire what we may call 'a geometrical culture for teaching'. We have deliberately abstained from listing specific geometrical topics, concepts, facts, methods, techniques, theorems, theories, etc. that teachers for such and such level should know or master. We have chosen to do so in order to avoid the syllabusitis trap referred to in §2, and not because it is impossible or unreasonable to indicate specimens of this sort – for instance, it would be fairly 'safe' to require all teachers of geometry to know the area of a circle, and (the content of) the Pythagorean theorem. However, it would trivialise and distort the discussion of teachers' knowledge and view of geometry if we spent our effort on establishing a syllabus for their education.

Once the question of which dimensions of geometry should be considered essential for teachers' perception of the field at a given teaching level has been settled, the specification and implementation of these dimensions into concrete curricula and programmes can be carried out in infinitely many valid ways. Which way should be chosen in a given context depends so strongly on the actual situation, its boundary conditions and circumstances, that it does not make much sense to attempt to discuss or reach agreement concerning the design of 'canonical' curricula for pre-service or in-service

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teacher-training. Hence, it should not be taken for granted that the geometry packages designed for pre-service preparation of teachers should be restricted to containing only the classical subject matter which always used to be included.

5. Knowledge and views of processes of learning geometry

In the previous section we discussed teachers' knowledge and views of geometry as a *mathematical topic*. It is now time to turn our attention to the kinds of knowledge and views that teachers should have of the *processes of learning* geometry, at different educational levels.

Firstly, teachers' knowledge of various ways in which geometrical experiences can be gained, geometrical notions and concepts can be established, and knowledge, understanding and ability be acquired, will, roughly speaking, be specialisations to geometry of their knowledge of the ways in which mathematical concepts and insights in general are acquired and established. This is an area in which research in mathematics education has been very active during the last couple of decades, and continues to be so. Lots of reports and papers have been published to this effect. As this is not the place to present or review recent research in the area and its outcomes at a general level, let us confine ourselves to pointing out just one major finding on which there is universal agreement in the mathematics education community, no matter whether the issue is considered from some form of a so-called constructivist position or not. We should like to propose that this finding be called *The First Main Finding of the Didactics of Mathematics:*

When a pupil or student engages in learning mathematics, the specific nature, content, range, and flavour of a mathematical notion or concept that he is acquiring or building up are greatly influenced, if not determined, by the set of domains *in which that notion or concept is anchored and imbedded for that particular person.*

In other words, mathematical notions and concepts and their interrelations are immersed in, and are hence strongly coloured by, pupils' and students' personal worlds of experience. This implies that the meaning and the interpretation which an individual assigns to a mathematical notion or concept are marked by that individual's entire set of mathematical and extra-mathematical experiences. This is true not only for notions and concepts but also for the interpretation and the range of validity of mathematical statements (propositions and theorems).

Clearly, the general finding just described should be included in the luggage of any mathematics teacher. In the nature of the matter, it pertains to geometry too. However, what lessons can be learnt from this finding that are specific to geometry and its teaching and learning?

In contrast to what is the case with many other mathematical topics and branches, characterised primarily by abstractness and generality, geometry

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is linked to physical objects, physical space, and the perception thereof, in a multitude of different manners. So, any individual who is learning geometry, in whatever form and at whichever level, already possesses an infinitely rich fund of geometrical knowledge and experiences ('correct' or 'non correct' that is not important here) which are of a mathematical nature, albeit not necessarily expressed or represented in a mathematical language or framework. This implies that an individual's learning of 'new' topics in geometry, in a mathematical formulation, will inevitably be confronted with the geometrical intuition, insight, knowledge, and experience already present in the pupil's/student's world of experience. Naturally, this constitutes a wealth of opportunities but, as a matter of fact, significant obstacles as well. It is important that teachers are aware of both the opportunities and the obstacles to an extent which allows them to take advantage of the former and to counteract, avoid or circumvent the latter.

As far as the opportunities are concerned, the learner's everyday geometrical experience provides the main source and background for his forming geometrical notions and concepts, in a more formal mathematical setting. The same is true with geometrical propositions, methods and techniques which can often be established in a dialectic interplay with corresponding propositions, methods and techniques residing in the everyday geometry of viewing, drawing, folding, cutting and pasting, constructing, measuring, moving, and so forth, with physical objects, shapes and phenomena. Furthermore, conjectures about properties of formal or abstract geometrical objects can be obtained or supported by the learner's exploration of the geometrical objects of his experiental world. Hypotheses may be subjected to preliminary empirical testing before one embarks on attempts at more formal derivations or deductions. Any learning of formal geometry will have to imply a reconciliation in the learner's mind of the formal representation and understanding of the topic at issue with that learner's experiental geometrical universe. This reconciliation requires the development of plane and spatial intuition along with the investigation of formal geometry.

To sum up, the epistemology of geometry (i.e. the nature, role and status of geometry and geometrical knowledge) becomes an urgent issue in the learning of geometry as soon as the conception of geometry as a solely empirical discipline is abandoned. The learning and understanding of 'post-empirical' geometry can hardly take place unless epistemological issues are brought into the open and dealt with in the learning process. Henceforth, any meaningful teaching of geometry will have to strive to create or at least help this reconciliation between empirical and theoretical geometry in the minds of each of the pupils or students to whom the teaching is being given. However, further reflections on this theme belong to the next section of this chapter.

From the 'Main Finding' referred to above, we learn that the content, meaning, and range of a geometrical notion, concept or statement, learnt

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or constructed by a learner, are (co-)determined by the specific experiental domains to which it is connected for that particular learner. The richer the set of domain connections, the more comprehensive and multi-faceted the concept, notion, or statement – and conversely. In other words, for pupils and students to develop a certain geometrical knowledge, that knowledge is likely to be exactly as profound and extensive as its explicit experiental foundation allows for.

In a way, what we have just said is simply a reflection of the point made in §4 that one of the essential dimensions of geometry is that it is a natural and a design science, with a strong empirical basis. None the less, if the ambition is that the learner's understanding and perception of geometry should reach beyond its immediate naturalistic and empirical dimension, it is exactly this very dimension which can create serious obstacles to the learning of geometry.

Probably the key obstacle is the potential equating of geometry with ontology (i.e. the constituents of reality, primarily physical reality). The majority of concepts in theoretical (Euclidean) geometry – whether synthetic or analytic – have close counterparts in everyday reality. This is true with, say, line, point, circle, triangle, prism, pyramid, sphere, ball, plane, box, pentagon, length, area, volume, angle, etc. In fact, as we all know, such geometrical concepts were initially formed as abstractions from corresponding 'everyday' concepts. Of course, the abstraction process implies an idealisation as well, for instance when, in axiomatic geometry, we think and speak of lines and planes as infinite, of points as having zero length and area, and so forth, but still the formal geometrical concepts inherit large parts of their meaning from the corresponding 'real' ones. This intimate correspondence does not only encompass concepts but also properties and propositions (although, and that is important, not the *verification of propositions*). In general, properties and propositions in Euclidean geometrical theory can be interpreted almost immediately as properties of and propositions concerning (suitably idealised) geometrical objects of the real world. Now, the obstacle arises primarily in two contexts.

Firstly, as regards the methods of verification of assertions in the theoretical versus the empirical domain. In the theoretical domain, verification is based on some form of reasoning, the specific format of which depends on the nature of the geometrical theory involved. (In axiomatic geometry à la Euclid, reasoning is based on deduction from the axioms and the givens; in analytic geometry it is based on the real number system and its properties.) In contrast, verification in the empirical domain is primarily based on empirical inspection followed by induction (in the philosophical sense, of course). It is well known that many learners of geometry have severe difficulties in understanding and appreciating this difference and in accepting the mode of verification adopted in geometrical theory. Why prove assertions the correctness of which are intuitively evident to any observer,

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or can be substantiated convincingly by a suitable number of empirical tests? To understand and accept that the purposes and goals, and hence the methods of justification, in the two domains are different is non-trivial and requires a substantial intellectual effort on the part of the learner. We believe, therefore, that it is highly important that mathematics teachers have insights into the origin and nature of these learning obstacles, and into ways in which they can be overcome. A possible starting point for this could be, for example, to consider the problem of obtaining certainty as regards general statements, like: *the sum of the angles of a plane triangle is always* 180° , or: *the medians of a triangle always intersect each other in one point.* These are claims which cannot be justified universally on empirical grounds only. Similarly with the fact that geometry (at higher levels, that's true) offers 'theorems of complete classification' and 'theorems of impossibility', such as: *the only convex regular polyhedra that exist (in 3-space) are the five platonic solids*, or: *it is impossible to trisect any arbitrary, given angle by means of ruler and compass in Euclid's sense.* Claims of this kind may form a point of departure from which the very nature and tasks of geometry can be considered.

Secondly, if the objects of Euclidean geometry are being assigned a direct status in physical reality, it is very difficult to come to grips with other sorts of geometry in which, say, the objects called "points" and "lines" are of a different kind to those in empirically based Euclidean geometry, and where notions like "parallel" and "perpendicular" have unexpected meanings and implications, as is the case, for instance, in spherical geometry. Evidently, this difficulty is reinforced if we enter the realm of absolute geometry, inhabited by completely abstract objects which might as well be called "tables", "chairs" and "beer mugs", to quote Hilbert. The really subtle thing is that when this non-naturalistic status of the objects of Euclidean geometry (of whichever brand) is emphasised, the problem pointed out here not only remains but is aggravated by the fact that the ontological status of naturalistic Euclidean geometry 'butts in', so to speak, with a competing interpretation of the objects and phenomena encountered in the geometrical theory at issue.

Again, if pupils and students are to acquire understanding and insight into these intrinsically difficult aspects of geometry, these have to be rooted in experiences which learners gain from exploring and investigating a variety of geometrical domains, both empirical and theoretical. And this takes time. If the learning process, for one reason or another, does not require the learner to explore and investigate a sufficient number of representative domains, then that learner's knowledge and view of geometry are fairly certain to become either superficial (geometry is a sort of strange game with no external meaning) or to suffer from severe misconceptions.

In conclusion, for a mathematics teacher to become able to adequately teach geometry to his pupils/students, it is necessary that he possesses

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knowledge of how the learning of geometry may take place and of the pitfalls that might impede such learning. It is a *conditio sine qua non* for meaningful and fruitful learning of post-elementary geometry, to establish viable links in the mind and experience of the individual learner, between geometry as a naturalistic discipline and geometry as some form of theory. This process requires a lot of time – and, naturally, learning opportunities – to be developed. The Scylla of viewing geometry as merely a naturalistic, empirical discipline, and the Carybdis of reducing geometry to just a formal game played according to sets of arbitrary rules, are always present as a challenge to the learning and teaching of geometry. Teachers of geometry have to be continually engaged in counteracting them. This takes us to the next section.

6. Rich teaching-learning environments in and with geometry

On the basis of the conclusions we have obtained in the previous sections, this section will consider how teachers should be able to design and orchestrate rich teaching-learning environments that allow pupils and students to acquire knowledge, insight, and ability concerning those dimensions of geometry which are on the agenda at their level of education. As has been the case so often in this chapter, much of what is going to be said addresses teaching-learning environments in mathematics education in general, with geometry as a special case. However, a number of deliberations that may be seen as specific to geometry will be offered as well.

First of all, teachers have to deal with the overall purposes of, and the specific objectives for, the teaching of geometry to a given category of pupils or students. The complexity of geometry as outlined in §4, as well as of the learning processes sketched in §5, gives rise to a large variety of different purposes and goals – not necessarily mutually compatible – that might be pursued at a given teaching level and in a given context. The point is not only that teachers will have to know the official goals for the teaching and learning of geometry in a given curriculum, as established by the relevant authorities. More importantly, teachers should reflect, with critical analysis, on these goals as well as on the goals which they themselves would like to put forward and pursue as professionals in the teaching of mathematics. Moreover, they should reflect on the degree of compatibility between these categories of goals, and finally make their own decisions, within the room of 'goal autonomy' which is left to them by the curriculum.

Next, it follows from the discussion in §5 that the acquisition of geometrical knowledge, insight and ability depends strongly on the specific domains in which the geometrical experiences of the learner are gained and rooted. Therefore the most significant task of the teacher of geometry is to stage teaching-learning environments, situations, and activities

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in which the learner can study, explore and investigate the objects, phenomena, problems, properties, and structures of a broad spectrum of geometrical situations and 'worlds'. These teaching-learning contexts, and the domains they invoke, should be as rich and many-sided as possible. Sometimes pupils/students should work on their own, individually or in small groups, exercising free exploration of, typically, an object world or a problem situation. Sometimes, pupils should be engaged in activities designed and guided by the teacher, and sometimes they should be recipients of stimulation, information, or explanation, supplied by the teacher, or by various sorts of materials or media, including information technology. Sometimes work should take place in the classroom, sometimes in out-of-school activities. At any circumstance, the teacher needs to be able to design curricula and to construct, select and implement written, concrete or electronic teaching materials suitable for the specific teaching purposes on the agenda.

The point made in the preceding paragraph is not special to geometry but concerns the teaching of mathematics in general. However, it is especially important with respect to geometry, because of the fundamental duality between geometry as a natural and design science and as a theoretical discipline. The reconciliation of these two dimensions of geometry requires a multitude of experiences gained from work in both dimensions. At the primary level, the teacher should be able to design and organise teaching-learning contexts where the pupils can explore concrete, tactile physical objects; some of the latter exist in nature, others are artefacts presented to the pupils, and still others are constructed, under teacher guidance, by the pupils themselves of paper, cardboard, wood, and so forth. Some objects should be solid, others transformable in various manners. Geometrical objects created by computer graphics should be included in the teaching-learning environment as well. The pupils should investigate the shapes, sizes, and properties of the objects they study, solve problems, pose and answer questions, etc.

At lower secondary level, the sorts of activities mentioned for the primary level – many of which are equally relevant at higher levels *mutatis mutandis* – should be complemented with teaching-learning situations where relations between geometrical experience of physical space and physical objects, and formal geometrical theory can be established, and the interplay between them investigated by pupils. This requires activities through which the pupils can study and investigate the conceptual aspects of introductory geometrical theory (of whatever kind), with a focus on the geometrical concepts as resulting from idealisations of certain features of everyday objects and phenomena. As this concept formation represents a step of far-reaching impact on the foundation of the geometrical knowledge and insight of the individual pupil, it is essential that the teaching-learning situations are

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sufficiently many and sufficiently rich to allow for this foundation to be extensive and firm. Teachers need to be able to flexibly create such situations in accordance with their pupils' needs, backgrounds and situations.

While the teaching-learning contexts at the lower secondary level are to emphasise the geometrisation of aspects of pupils' physical environment, the relationship between physical reality and geometric formalism is often reversed at the upper secondary level. That is to say that *in addition* to what has been proposed for the primary and lower secondary levels, quite a few teaching-learning situations for the upper secondary level should provide opportunities for students to go in the opposite direction, i.e. investigate to which extent it makes sense to interpret concepts, phenomena, and propositions from theoretical geometry in terms of physical reality. This includes inquiries into the range of correspondence between concepts and propositions of geometric theory, and physical reality. Also, students should be engaged in activities that focus on the validation of geometrical assertions within the formal frameworks at issue. Again, it is a task for the teacher to design and orchestrate teaching-learning environments that can support students' acquisition of insight into the nature of theoretical geometry and its interplay with physical reality. Numerous examples of teaching-learning environments are given in previous chapters (e.g. 2, 4, and 5) of this book.

As finally regards the teaching of geometry at tertiary levels, which typically focusses on various forms of abstract and general geometrical theory, the critical task for the teacher is to provide encouragement, stimulus and environments for students so as to allow them to examine the relationship between advanced geometrical theory and elementary geometrical theory. Special emphasis should be placed on recalling and interpreting concepts and propositions from elementary geometry in advanced settings. For instance, which aspects of the elementary concept of angle (or plane) are preserved in the definition of angles (or hyperplanes) in vector spaces with inner products, and which are not? Or, to what extent does advanced projective geometry generalise the geometry of, say, perspective constructions? And to what extent does projective geometry give rise to phenomena which have no counterparts in elementary geometry of perspectivity? Of course, the reasoning and deduction modes, and their basis, in different geometries also deserve close attention by teachers and students at tertiary levels. Far too often, in tertiary-mathematics teaching, the bridging of the interpretation gap between advanced theories of geometry and elementary geometry is left to students' own endeavour and effort, which in actual practice amounts to disregarding the need for such links. How can we imagine future teachers of mathematics to see and establish such links in their own teaching of geometry if they have never had an opportunity to encounter these links as students themselves? Probably much of the problem stems

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from the fact that a non-negligible number of tertiary-mathematics professors do not take teaching seriously but confine themselves to lecturing. The time seems to be ripe for a general call on university mathematicians to invest not only time but also intellectual effort in quality teaching.

For all the levels considered, three points are of particular significance in relation to the teaching of geometry.

• Because of the special nature of geometry as a mathematical discipline, geometry is unique in providing opportunities for "getting much out of little", didactically and pedagogically speaking. Probably any geometrical situation, however elementary, gives rise to a rich variety of possibilities for exploration, posing of questions and problems ("what if..?", "could it happen that ...?"), formulation of conjectures, experimentation with special cases, attempts at convincing others, at proving, and so on and so forth.

• Because of its age-old position as a natural and design science, geometry is intimately linked to human society, culture, history, science, technology, philosophy, arts and crafts. The teaching of geometry, therefore, offers ample opportunity to uncover and relate mathematical ideas to human and social activity.

• Probably, there is no better place than geometry for illucidating and discussing the concept and role of proof and proving in mathematics. All possible notions and variants of reasoning, justifcation and proof are present in opulent measures in the context of geometry. The entire spectrum of seeing, believing, following, or accepting a statement or a line of thought, of being convinced, persuaded, intimidated etc. can be encountered and investigated in geometrical settings (see Chapter 2).

In their endeavours to stage rich teaching- learning environments, teachers should be able to take advantage of these three features of geometry, adapted to the level, context, and situation in question.

Finally, as far as the role of information technology is concerned, a remark in conclusion is warranted. There is no doubt that for all the levels we have been considering, information technology offers powerful and splendid tools for the furthering and support of the development of geometrical experience and intuition, and the investigation of geometrical worlds, for instance through the opportunities to formulate and test hypotheses and conjectures (see Chapter 4). However, it is important to emphasise that this does not imply that information technology can *replace* the real world of natural and man-made objects. The fundamental insight that geometry *also* is to do with the real world cannot be experientally substantiated without systematic geometrical experiences with and from the real world.

7. Teacher assessment in geometry

A separate chapter (Chapter 8) in this book is devoted to the question of assessment in geometry. In this context, therefore, we shall confine

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ourselves to considering – briefly – those aspects of assessment in geometry that are to do with the qualifications and education of teachers.

If we agree that teachers should not concentrate only on 'delivering' their own teaching but also on paying attention to their pupils'/students' learning, it follows that teachers should be able to observe, analyse, and monitor pupils' and student's learning, and to offer guidance and advice concerning the adjustment of learning strategies and behavior, for individual study as well as for classroom work. This requires teachers to be in possession of a variety of assessment modes that are suitable for different purposes and goals in relation to geometry, among which examinations and promotion tests occupy only a position of secondary importance, if viewed from a didactical perspective (Niss, [4]).

This implies that one main purpose of teacher assessment in geometry is to procure and provide *information* to the learners concerning the progress they make and the problems they have in acquiring geometrical knowledge, insight and ability. Moreover, such information should also serve to enable the learner to monitor and assess his own learning of geometry. Another main purpose is to place and *guide*, on the basis of the information obtained, the learner in learning situations and activities which can help to underpin and develop knowledge and ability in those aspects of geometry in which there is a particular need for it.

In accordance with the discussion in §5, teachers should, in their assessment of an individual pupil's/student's learning of geometry, pay particular attention to:

(a) the extent to which he has overcome the epistemological obstacles caused by the dual nature of geometry (as a natural and design science and as a formal discipline, respectively, cf. §4 and 5) as far as the acquisition of geometrical notions, concepts, and propositions are concerned;

(b) his ability to interpret matters and situations originating in one geometrical world in terms of another such world;

(c) his ability to reason within a given geometrical universe;

(d) his ability to solve pure and applied geometrical problems;

(e) his ability to pose questions and problems, and to formulate hypotheses and conjectures.

In other words, the education of teachers with respect to the assessment of geometry should focus on ways in which these components of the learner's geometrical knowledge and ability can be identified, diagnosed, analysed (and ultimately improved). As to a discussion of possible means and tools for that purpose, we refer to Chapter 8. Suffice it, here, to say that timed written tests cannot stand alone in the pursuit of that purpose.

8. Ways to develop a competence to teach geometry

In the previous sections, our attention has been focussed on ideals rather than on the everyday situations, circumstances, and constraints of the real

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world. Now, an obvious question poses itself fairly strongly: To what extent, and in what ways, is it possible to pursue these ideals in actual practice? More specifically, can we ensure that teachers of geometry acquire knowledge and views of geometry as a mathematical topic and of the processes of learning geometry, that they become able to stage rich teaching-learning environments and situations in and within geometry, and that they become able to assess their pupils' or students' learning of geometry so as to identify, monitor, and guide the learning processes and strategies of these pupils and students, in accordance with the perspectives outlined in the previous sections? Unfortunately, the answer to this question, as it stands, is probably "no", in general. For the answer to the question to be "yes", Utopia had to be invoked. But less will do. Instead, we may ask what could be done to approach the ideals sketched in this chapter. The remainder of this chapter is devoted to this issue. A remark in advance: clearly, in view of the immense diversity of circumstances across and within countries and institutions, it is not reasonable to expect that it is possible to come up with specific, concrete ideas and suggestions which can be implemented right away in any context whatsoever.

In §3 it was stated that for no teacher, irrespective of the educational level at which he teaches, will his pre-service preparation be sufficient for his entire career as a teacher of mathematics. This is true also for the preparation to teach geometry. Therefore, continual in-service education and professional development of the individual teacher becomes a necessity if we (i.e. society and the mathematics education community) want teachers to be able to approach the ideals discussed here. However, successful in-service professional development has to be built on a fairly solid pre-service basis of subject matter knowledge and views combined with didactical and pedagogical knowledge and insight. As regards geometry, this implies that all the main aspects dealt with in this chapter should, *in principle, be on the agenda* of any pre-service preparation, even if the depth to which they can be treated will necessarily differ from level to level and from place to place simply because the conditions and problems vary greatly with country, teaching level, and so forth.

In many countries it is customary to educate primary school teachers to obtain only a rudimentary knowledge of mathematics beyond arithmetic. It is not unusual that primary school teachers of mathematics have a knowledge of geometry which can only be characterised as being below the minimum that any meaningful teaching of geometry does require. Although this chapter is not the place to discuss the overall structure and organisation of teacher-training programmes for primary school teachers (in most countries this is a highly political and socio-economic issue), it seems that in quite a few places even a small (absolute) increment in the geometrical content of these programmes will result in a substantial, and much needed, relative increase in primary school teachers' geometrical knowledge and

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an improvement in their prerequisites for teaching geometry. Put differently, there is a potential for development of competence here which is not fully exploited in many places. Moreover, such a solidification of their pre-service backgrounds would provide a basis for primary school teachers to later participate in, and benefit from, the kinds of in-service activities on the teaching of geometry which will be discussed below.

As to the secondary level, the diversity of pre-service teacher-training programmes, with respect to structure and content, among countries, institutions and so forth is even greater than on the primary level. In some countries secondary school teachers are prepared to become specialists in a few subjects, for instance by being required to have a university Master's degree in those subjects, whereas other countries educate teachers to teach a broad spectrum of subjects for which their preparation is then, necessarily, less subject-specific and intensive. However, it seems fair to summarise the situation as follows (in very gross terms, of course; several counter-examples do exist): in programmes emphasising subject-matter specialisation, teachers of mathematics are likely to be given some, although in general not substantial, preparation in geometry as a discipline, but seldom much in the didactics and pedagogy of geometry. In programmes emphasising a general preparation for the profession of secondary teacher, teaching in general tends to be in focus, and the mathematical part of the programme is usually less manifest and comprehensive, and hence the same is true for geometry, both as a mathematical and as a teaching topic. No matter which type of in-service teacher-training programme we consider, there is reason to believe that there is room left for improvement with respect to the preparation of future teachers for the teaching of geometry, regarding either the subject-matter side or the didactical/pedagogical side, or both.

As finally regards the background of university lecturers and professors for teaching geometry, including geometry to future teachers, the average picture is that they are fairly well versed in geometry as a mathematical discipline, but rarely in geometry as a topic for teaching and learning. Here is clearly room for progress as far as the didactics and pedagogy of geometry (or of mathematics at large, for that matter!) are concerned.

Although it is evident that it is desirable to equip future teachers with as solid as possible a pre-service preparation concerning the dimensions of geometry, previous considerations in this chapter led us to conclude that even the most splendid pre-service programme should be followed up by continual in-service education and professional development of the individual teacher on whichever level. Such continual professional development can – and should – take a multitude of different forms and formats, depending on all the needs, boundary conditions, factors, circumstances and opportunities that prevail in a given context. So, the important thing is not so much *exactly* what happens, and how it happens, but rather that

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something happens that may serve the purpose of enlarging and enriching teachers' qualifications as teachers of geometry with respect to:

- geometry as a mathematical topic, in all its manifestations;
- processes of learning geometry, viewed from the perspectives of practice and research;
- environments, contexts, situations, and activities for the teaching and learning of geometry.

Just a few examples of ways in which the professional development of teachers of geometry can be staged are: in-service courses and workshops, summer schools, conferences, publication activities, local, regional or national informal networks or formal organisations of teachers and researchers working at various levels, teaching experiments, participation in development or research projects on the teaching and learning of geometry, international exchange of ideas and experiences. For this to be furthered, "let a thousand flowers bloom".

In-service activities which go across teaching levels and institutions seem particularly worthwhile and fruitful, not only because they serve as a platform for exchange of information and inspiration. For they also stimulate teachers on a given level to articulate and reflect on, and to re-consider, what they are doing, if they are to communicate about it to professionals in other segments of the educational system. One example of this is the IREMs in France in which teachers and researchers on various levels collaborate within a regional context (see Douady and Henry, [1]). Another example, addressing the tertiary level only, is the so-called *Undergraduate Mathematics Teaching* conferences (known as the *Nottingham Conferences*), in the UK, where, with the goal of raising critical consciousness, small groups of university teachers are expected to provide *written* arguments about, e.g., curricula, to criticise the work of other groups in writing, and so forth (Brian Griffiths, personal communication).

In conclusion, once again it is true that much of what has been said in this section with respect to geometry can be considered as a special case of what could be said of mathematics at large. This is because geometry is a part of mathematics, yet a very special one, and because geometry is taught by teachers of mathematics. To those teachers, and to their pupils or students, and to the mathematics education community at large, geometry should be taught for the same reasons as mathematics as a whole is taught: in order to help the recipients be able to better understand and cope with life in society, culture, and nature.

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10. THE WAY AHEAD

Vinicio Villani on behalf of the International Program Committee

Wer von uns würde nicht gern den Schleier lüften, unter dem die Zukunft verborgen liegt, um einen Blick zu werfen auf die bevorstehenden Fortschritte unserer Wissenschaft und in die Geheimnisse ihrer Entwicklung während der künftigen Jahrhunderte! Welche besonderen Ziele werden es sein, denen die führenden mathematischen Geister der kommenden Geschlächter nachstreben? (David Hilbert, *Mathematische Probleme.* Vortrag gehalten auf dem Internationalen Mathematikerkongress zu Paris, 1900)[1]

Not even Hilbert, one of the most far-sighted mathematicians of the last century, possessed a crystal ball with which to read the future developments of mathematical research! The solutions found some decades later for several of the 23 world-famous problems he posed to the mathematical community in the speech quoted above, have been inherently different from Hilbert's own guesses. Just think of the unexpected 'negative' answers by Paul Cohen and by Kurt Gödel to his first two problems (on the "Continuum Hypothesis" and on the "Consistency of Arithmetic Axioms").

Hazardous as it is to try to predict the future of mathematical research, it may be even more hazardous to try to predict the future of mathematics education, since this is a field that is subject to an even wider variety of factors, both intrinsic and extrinsic to the specific discipline. Who, for instance, could guess only 50 years ago that straightedge and compass would be ousted from all the professional workplaces of engineers and technicians by CAD graphic tools? Or that there would be a rebellion (later on at least partly softened) against the teaching of geometry in Euclidean style? Or, further, that advanced geometric techniques would become an essential tool for an improvement of the traditional X-ray images by stratigraphic methods?

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[1] Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during the future centuries! What particular goals will there be toward which the leading mathematical spirits of the coming generations will strive? (David Hilbert, *Mathematical Problems.* Address delivered at the International Congress of Mathematicians in Paris, 1900).

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In spite of the difficulties in making long term forecasts, we cannot stay inactive in our teaching practice, while time passes us by.

Changes in geometry teaching are imposed not only by new insights in the foundations of geometry and by an increasing awareness of the central role of appropriate teaching methodologies, but also by the changes in the expectations of society, by the changing needs at workplaces, and by the steady progress of science and technology.

Therefore it is necessary and urgent to look for a connecting thread which gives us a guideline from the recent past to the near future. In this spirit, we are now going to enumerate some specific findings, which emerge from the previous chapters and more generally from international educational literature concerning geometry. But we are conscious that many questions raised e.g. in the ICMI Discussion Document (reprinted in the Appendix) still remain open, and that several adjustments and changes of route may be needed in passing from general statements to concrete actions for an improvement of the teaching of geometry.

1. One of the peculiarities of geometry is the variety of its aspects (as pointed out in Section 2 of the ICMI Discussion Document), as well as the variety of the possible ways of its teaching, depending on the priorities of the goals we want to attain and on the type of pupils or students we address. What is certain and well documented in the literature, is that at every educational level, from Kindergarten on, there exist parts of geometry that can (and should) be taught and learnt successfully (see e.g. Chapters 2 and 5).

We are aware that in order to achieve a global understanding of the main geometric facts, concepts, procedures, strategies, it will be necessary to come back again and again to the same topics at several stages, with a spiral-like process. But this fact should not be considered with a sense of frustration. Only long-term effects really matter in a teaching-learning process. It would be a serious error to avoid dealing with a certain topic under the pseudo-pretext that the same topic will be dealt with, in a more general and more systematic context, at a later school level. This behaviour would be quite harmful, since pupils would miss the preliminary knowledge, needed as a starting point at the subsequent level. As R. Thom has already pointed out at ICME 2: according to Haeckel's law of recapitulation (asserting that ontogenesis recapitulates phylogenesis) *pedagogy must strive to recreate the fundamental experiences that, from the dawn of historic time, have given rise to mathematical entities.*(2)

Thom's statement may be reworded as follows: if the teaching-learning process has to be meaningful for learners and not just a formal game, one must not skip any intermediate part of the path which leads gradually from concrete and intuitive aspects to abstract, deductive geometry.

(2) *Developments in Mathematical Education*; Proc. of the Second International Congress on Mathematical Education, A.G. Howson Ed. (Cambridge Univ. Press, 1973). This whole paper is still relevant to us now, and especially to the concerns of this book.

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2. To build a curriculum is a very difficult and demanding task that must involve mathematicians, teachers, educational experts having a deep knowledge of the real needs and expectations of the specific community for which the curriculum has to be designed (see Chapter 7). The failures of many mathematics curricula are still before everybody's eyes. These curricula were imposed by a top-down process, without taking account of the opinions and the experiences of working teachers (see Chapter 6). No wonder that failures occur also when curricula are transplanted *en bloc* from one country to another, typically from a dominating country to the area of its influence in other parts of the world.

Any curriculum, in order to be effective, must have an inner coherence, otherwise the learning process descends into a simple rote-learning of a bazaar of unrelated notions. This remark applies to curricula in every subject area, but is especially true for geometry, due to the great variety of its aspects. We might suggest the analogy with a bus journey: if you leave school at any particular stage, it is desirable that you have enjoyed the journey so far, and may hope to rejoin it later. Therefore curriculum planners must make coherent choices concerning goals, contents and methods; textbook writers must make further didactical choices while implementing the prescribed curricula, and teachers must make additional choices, and state priorities, at the moment of the didactical transposition. In making these choices and in stating these priorities, the relevant variables to be considered are not only the school levels and the ages of pupils, but also cultural, linguistic, socioeconomic aspects, as well as feasibility, depending on the overall structure of the local educational system, teacher preparation, availability of technological resources, and so on.

For all these reasons, it would be improper to claim that it is possible to elaborate a geometry curriculum having universal validity. It is only possible to highlight some general aspects, on which there is a wide international consensus. However, some carefully formulated examples of curricula from five selected countries are presented with commentary in Chapter 7.

As already stated above, a curriculum must bear an internal coherence. Nevertheless some discontinuities are unavoidable in progressing from one stage to the next. For instance, when one passes from the early, naïve and reality-linked manipulative and visual phases to a more advanced and theoretical point of view, the "didactical contract" suddenly changes: pupils who until then were allowed to 'see' the geometric properties of a configuration from a drawing, must now begin to deduce these properties using logical argumentation. These discontinuities, in order to be effective and not to disorient pupils, must be thoroughly motivated, showing at each step that the new setting is an improvement on the former one. This applies of course also to the relationships between synthetic and analytic methods.

3. At primary level, geometry should not be limited only to some terminology. Instead, it would be desirable that, from the earliest grades,

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geometry teaching should help pupils to improve their spatial abilities, and foster experiences in the measuring of lengths, areas and volumes so as to gradually make them acquire confidence in the use of international (metric) units. Practice with ruler, compass and protractor is still desirable, regardless of the possible use of computing aids. Moreover, from primary school on, it is quite possible to involve pupils in classification activities, e.g. identifying symmetries and regularities of simple geometric shapes.

At middle school (age ≤ 15), geometry should not be limited only to rote-learning of some formulae for areas and volumes. At this school level it would be desirable that pupils are made aware of the main properties of isometries and similarities, and that they experience some first instances of deductive reasoning (e.g. a proof of the classical theorem of Pythagoras).

At upper secondary school (age ≥ 15), geometry should not be limited only to formal "two column proofs" (3), nor only to linear algebra. It would be desirable that pupils came to know at least conic sections, as well as some significant applications of geometry. Indeed, computer graphics could save much time in displaying properties of conics.

Further suggestions for less traditional activities for all grades, from the beginning of primary school until the end of upper secondary school may be found in several sections of this book (*e.g.* in Chapters 2,3,4,5).

4. We may also without any doubt assert that geometry should not deal only with 2-dimensional shapes. And moreover that 2-dimensional geometry should not deal only with the "microspace" of a sheet of paper or a page of a book (see Chapters 2 and 5). On the contrary, and especially during the first school grades, the starting point for geometry should be a careful observation of the 3-dimensional reality in which we live. And this attention to 3-dimensional situations should continue even in later grades, together with a focus on the relationships between 3-dimensional space and its 2-dimensional plane representations: the objects 'as they are' and 'as they appear' (on the retina of our eyes, on a sheet of paper, on a computer or TV screen). In this context it is natural to deal not only with metric aspects of geometry, but to consider also affine properties of the plane, as well as parallel projections from space onto a plane. Moreover one might consider (at least in a naïve form) central projections, i.e. perspectivities.

5. In a certain sense, teaching methods are even more important than content. And it is also more difficult to improve them!

During the last four decades, criticism of the traditional way of teaching synthetic (Euclidean) geometry in upper grades has been directed towards the style of the presentation of the whole construction as an untouchable end-product, rather than towards the content itself.

Nowadays it is generally agreed that the rote-learning of ready-made proofs has a poor educational effect and does not achieve the desired goal,

(3) See footnote at page 30.

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which is to familiarize pupils with hypothetical-deductive reasoning in different (not only geometrical) contexts.

We believe that at upper secondary level pupils should get an idea of the structure of a deductive theory, since this is the main characteristic, which distinguishes mathematics from experimental sciences. But this does not mean that the teaching of 'deductive geometry' should start with a set of axioms, nor that every statement should be proved. With excessive systematization we risk killing the interest of pupils; this would mean achieving precisely the opposite of the desired educational effect.

Therefore, at least at the beginning of secondary school (say in grades 7-10, but often also later on) it seems advisable to use only "local" deductions, that are within reach of pupils and that therefore allow their active involvement. Of course, what should be made clear even at this stage, is that every proof relies on some assumptions that must be explicitly stated, and that are taken for granted. These assumptions may be seen as a provisional (and redundant) set of axioms.

In selecting the theorems to be proved, non-intuitive, surprising statements should be privileged, in order to show the power of reasoning over mere physical experience (*e.g.*: The sum of the angles of a triangle, Pythagoras' Theorem, Properties of the Euler Line in a triangle, Incommensurability of side and diagonal of a square). Such items could serve as 'staging posts' for a curriculum that envisages geometry as a journey, according to the 'bus' metaphor of §2.

Finally, 'theorem proving' should not be confined to geometry. Even in arithmetic, algebra and probability, there are many opportunities for proving easy but significant statements. Of course, the style of proofs is rather different in these contexts. What is peculiar to geometric proofs is the role of visual intuition, which is why it is so difficult to give a totally rigorous proof of anything in geometry (and why New Math turned to algebra instead).

6. Applications of geometry (both the impressive new ones (Chapter 1, Section I) and the well known classical ones (Chapter 1, Section II) are doubtless fascinating and therefore they should be emphasized. But one must avoid the danger of an excessive fragmentation. Just as an excessive systematization may kill the interest of the pupils, so an excessive fragmentation may be dispersive and inconclusive. The applications to be privileged are therefore those which best fit into a global, cultural and educational perspective of the whole curriculum.

Moreover, one should always keep in mind the recommendation of the Cockcroft Report(4):

(4) *Mathematics Counts*, HMSO, London, 1982.

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It should be a fundamental principle that no topic should be included [in a mathematics course] unless it can be developed sufficiently for it to be applied in ways which the pupils can understand.

7. There is now a wide convergence of opinions in the international mathematical community that geometry, after years of neglect, should be re-vitalized in its manifold aspects at all school levels.

But this is difficult to put into practice and it appears utopian, and maybe even undesirable, to believe that geometry may again come to monopolize nearly half of the time globally devoted to mathematics, as was the case in many countries until 50 years ago.

A way to overcome the consequent time-shortage is to lead geometry out of its isolation. This may be achieved for instance by familiarizing pupils with 'visual thinking' even in arithmetic, in algebra, in calculus, in discrete mathematics, in statistics and so on, using – whenever possible – sketches, drawings, graphs, diagrams and geometric terminology (e.g. distance, orthogonality, etc). It should become a routine habit of teachers, to ask pupils for a geometric interpretation of any kind of mathematical problems! It may also be useful to stress the links between geometry and other disciplines, e.g. physics, geography, technical drawing or even philosophy.

8. The use of computers may have a profound effect on the teaching of geometry from two points of view, which in a certain sense are complementary to each other.

(a) On the one hand the use of specific didactical software such as Logo, Cabri or Sketchpad can greatly improve the teaching-learning of geometry. At the same time, the use of this kind of software implies a deep rethinking of the role of activities such as conjecturing, arguing, proving.

(b) On the other hand, a deeper understanding of the graphical and computational aspects of geometry is needed in order to be able to use, with awareness, professional CAD software or other similar computer facilities. Such packages may also be used by skilful teachers as tools and environments to conduct activities for pupils' learning of geometrical concepts and structures (see Chapter 4).

As far as computers are concerned, a serious problem arises. In many countries financial resources are not sufficient to endow schools with appropriate hardware and software. Hopefully this situation will improve in the future, thanks to the falling costs of these electronic supplies.

To teachers who do not yet have the opportunity to use computer facilities we suggest: always bear in mind that these electronic devices exist, and hence emphasize in your teaching those geometric skills that are likely to become more relevant in the perspective of the future professional life of your pupils.

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In any case, at least up to now, it seems inappropriate to consider the use of computers as the only possible technological aid for the teaching-learning of geometry. A familiarity of pupils with straightedge and compass or with 3-dimensional models of geometrical objects is still essential.

9. The in-service training of teachers and the training of prospective teachers is a key problem in order to pass from wishful thinking to concrete action (see Chapter 9). For teaching any topic adequately, one must know it thoroughly; actually one should know much more than one teaches, and in greater depth. But it would be an illusion to think that one can master all the knowledge before starting to teach; the very act of teaching helps one to acquire both depth and breadth.

Unfortunately a large number of in-service teachers have received in their time a training in geometry which is poor and both quantitatively and qualitatively inadequate.

As for the preparation of new teachers, there exists a big gap between the kind of geometry taught at present in most colleges and universities, and the geometry that prospective teachers are expected to teach to their future pupils.

Since at colleges and universities geometry courses deal mainly with linear algebra, algebraic topology or similar subjects, it is all too easy to predict that those who are students now, once they become teachers, will turn back to the unsatisfactory models of teaching elementary geometry, they experienced during their own pre-university studies. Teachers are more likely to teach as they were taught, than as they were told to teach!

The New Math movement failed to bridge the gap between school geometry and university geometry, mainly because it had impact only on pre-university curricula. It was based on the the assumption that certain parts of elementary geometry were 'dead' for contemporary advanced research. But this assumption, whether true or not, must not imply the exclusion of these parts of geometry from being taught in an appropriate cultural perspective. The time has now come to act at college and university level, stressing the fact that elementary, visual geometry is still alive as a permanent source of interest and inspiration, even for advanced research. To say it in the words used by D. Henderson at the Catania conference:

> Alive geometric reasoning is paying attention to meanings behind the formulas and words – meanings based on intuition, imagination, and experiences of the world around us.
>
> Alive geometric reasoning is knowing that geometric definitions, assumptions, etc. vary with the context and with the point of view.
>
> Alive geometric reasoning is making conjectures, searching for counterexamples, and developing connections.
>
> Alive geometric reasoning is always asking: why?

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At the same meeting, M. de Guzmán pointed out that human beings rely for about 90 % of their activities on visual perceptions ("*Homo est animal visuale*", he said).

Thus, the initial training of future (primary and secondary) school teachers should include courses having an authentic, truly geometric, visual content.

The authors of the present chapter urge the readers of this ICMI Study to sensitize their colleagues at colleges and universities to this crucial issue: a deep and critical knowledge of elementary geometry (including awareness of the role of visualization), foundations, non-Euclidean geometries, as well as applicational, epistemological, historical and didactical aspects, is absolutely necessary both for advanced research and for teaching. As Freudenthal once pointed out: "Everybody, not only teachers and textbook writers must have experienced true geometry at least once in their life."

10. A second key point for an authentic improvement of the teaching-learning process concerns assessment, an aspect that seems to be rather neglected in general, and especially for geometry. The reason is inherent in the very nature of the subject. In fact, one of the key features of geometric reasoning consists in the ability to translate statements from one code to another. Usually a problem, originally formulated in words from our natural language, must be transformed into a drawing or into a formula or (in later grades) into some algebraic equation. At the end of this process, the same path has to be retraced in the opposite direction: from the algebraic solution or from the numerical results, to the visual interpretation and finally to the answer expressed again in words from the natural language. This long and complex itinerary is not suitable for being assessed by multiple-choice tests or similar standard procedures.

Moreover the habit, prevailing in many countries, of using only written assessment procedures is inadequate for evaluating the comprehension of the ideas involved in a proof, or even to assess the ability to implement a geometric construction on a computer.

If assessment is limited to the knowledge of geometric terminology and to the use of formulae for areas and volumes or to the ability to solve algebraic equations, this implies a distortion of the true spirit of geometry.

It is well known that what teachers evaluate and the way in which they evaluate have a crucial influence on what pupils consider relevant and worth learning. Therefore assessment procedures ought to be shaped according to the aims of the desired teaching-learning process, not vice-versa.

Summing up, a variety of assessment methods should be used (see Chapter 8), including tests but also more articulated exercises and problems of a genuine geometric flavour, explorations of open situations, written reports about activities performed during a computer session or in connection with

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practical work, together with oral presentations and interviews where a real interaction between teacher and pupil takes place.

On the other side, we must recognise the cost of these non-standard assessment methods to the teacher, in time and energy.

All these problems can rarely be solved satisfactorily by individual teachers. They need to be addressed by those who administer the public examination system. Nevertheless, teachers may contribute to an improvement of the present situation through collective pressure on the administrators.

Whatever the kind of assessment used, it is important to state evaluation parameters as objectively as possible, but without sacrificing significant aspects to the myth of a total objectivity; for, that risks reducing the range of the aspects of the curriculum considered suitable for assessment.

11. As for textbooks, in many countries there is a noteworthy flourishing of initiatives. But all too often curricular innovations are simply juxtaposed with the previous traditional topics, thus generating repetitions and lack of an overall coherence. A recommendation for textbook writers therefore is, to avoid such an accumulative way of compiling texts, and to be more selective in the choice of a coherent path, that might be clearly intelligible to teachers and pupils.

At least for the most innovative topics, where a didactical tradition is still lacking, we suggest to complement textbooks with appropriate teachers' guides (as is already customary in several countries) and to provide specific publications for teacher training.

12. Every curricular change implies necessarily some difficulties inherent in the novelty of the subject, in poor teacher preparation, and in inadequate support of teaching aids. Therefore, in order that an innovation, in whatever field, be successful, a necessary condition is, that the teachers who are charged with the implementation be completely convinced that it is worthwhile doing, and hence be willing to bear the burden of long and demanding effort.

With this ICMI Study, we hope to have succeeded in highlighting at least some of the most relevant motivations in order to persuade our readers of the necessity for an improvement in the teaching of geometry at all school levels. Moreover, we hope to have given some useful hints indicating the direction in which to move.

We are confident that, after the difficulties of the present transition phase, geometry will resume the important place it deserves within mathematics education.

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The bibliography which follows consists of an alphabetical listing of volumes (and a very few articles) which have been chosen because they are classics, surveys, or show emerging trends in geometry and geometry education. When possible recent books are listed. Although no doubt everyone will find some favorite omitted, it is hoped that this list will serve as a representative sample for people who want to probe further into geometry's many aspects. This bibliography, thus, supplements the listings in the references of the individual contributions in this volume. Note that these books are not meant to be a list we recommend that either be acquired by individuals or libraries.

Although the bibliography lists items primarily in English, there are a sprinkling of titles in other languages. The readers are encouraged to explore books that are being written in languages other than their own native language so as to get a feel for what is going on in other countries. However, we recognize that every nation has its own equivalents of titles that are primarily in English in this list.

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APPENDIX

PERSPECTIVES ON THE TEACHING OF GEOMETRY FOR THE 21st CENTURY

Discussion Document for an ICMI Study

(*Reprinted from* L'Enseignement Mathématique, **40**, 345-357, 1994)

1. Why a study on geometry?

Geometry, considered as a tool for understanding, describing and interacting with the space in which we live, is perhaps the most intuitive, concrete and reality-linked part of mathematics. On the other hand geometry, as a discipline, rests on an extensive formalization process, which has been carried out for over 2000 years at increasing levels of rigour, abstraction and generality.

In recent years, research in geometry has been greatly stimulated by new ideas both from inside mathematics and from other disciplines, including computer science. At present, the enormous possibilities of computer graphics influence many aspects of our lives; in order to use these possibilities, a suitable visual education is needed.

Among mathematicians and mathematics educators there is a widespread agreement that, due to the manifold aspects of geometry, the teaching of geometry should start at an early age, and continue in appropriate forms throughout the whole mathematics curriculum. However, as soon as one tries to enter into details, opinions diverge on how to accomplish the task. There have been in the past (and there persist even now) strong disagreements about the aims, contents and methods for the teaching of geometry at various levels, from primary school to university.

Perhaps one of the main reasons for this situation is that geometry has so many aspects, and as a consequence there has not yet been found – and perhaps there does not exist at all – a simple, clean, linear, "hierarchical" path from the first beginnings to the more advanced achievements of geometry. Unlike what happens in arithmetic and algebra, even basic concepts in geometry, such as the notions of angle and distance, have to be reconsidered at different stages from different viewpoints.

Another problematic point concerns the role of proofs in geometry: relations between intuition, inductive and deductive proofs, age of students at which proofs can be introduced, and different levels of rigour and abstraction.

Thus the teaching of geometry is not at all an easy task. But instead of trying to face and overcome the obstacles arising in the teaching of geometry, actual school-practice in many countries has simply bypassed these obstacles, cutting out the more demanding parts, often without any replacement. For instance, three-dimensional geometry has almost disappeared or has been confined to a marginal role in the curricula in most countries.

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Starting from this analysis, and specifically considering the gap between the increasing importance of geometry for its own sake, as well as in research and in society, and the decline of its role in school curricula, ICMI feels that there is an urgent need for an international study, whose main aims are:

– To discuss the goals of the teaching of geometry at different school levels and according to different cultural traditions and environments.

– To identify important challenges and emerging trends for the future and to analyze their potential didactical impact.

– To exploit and implement new teaching methods.

2. Aspects of Geometry

The outstanding historical importance of geometry in the past, in particular as a prototype of an axiomatic theory, is so universally acknowledged that it deserves no further comment. Moreover, in the last century and specifically during the last decades, as Jean Dieudonné asserted at ICME 4 (Berkeley, 1980), Geometry "bursting out of its traditional narrow confines [...] has revealed its hidden powers and its extraordinary versatility and adaptability, thus becoming one of the most universal and useful tools in all parts of mathematics" (J. Dieudonné: The Universal Domination of Geometry, ZDM **13** (1) , p. 5-7 (1981)).

Actually, geometry includes so many different aspects, that it is hopeless (and maybe even useless) to write out a complete list of them. Here we mention only those aspects, which in our opinion are particularly relevant in view of their didactical implications:

– Geometry as *the science of space.* From its roots as a tool for describing and measuring figures, geometry has grown into a theory of ideas and methods by which we can construct and study idealized models of the physical world as well as of other real world phenomena. According to different points of view, we get euclidean, affine, descriptive, projective geometry, but also topology or non-euclidean and combinatorial geometries.

– Geometry as *a method for visual representations of concepts and processes* from other areas in mathematics and in other sciences; e. g. graphs and graph theory, diagrams of various kinds, histograms.

– Geometry as *a meeting point* between mathematics as a theory and mathematics as a model resource.

– Geometry as *a way of thinking and understanding* and, at a higher level, as *a formal theory.*

– Geometry as *a paradigmatic example for teaching deductive reasoning.*

– Geometry as *a tool in applications*, both traditional and innovative. The latter ones include e. g. computer graphics, image processing and image manipulation, pattern recognition, robotics, operations research.

Another distinction should be made with respect to several different approaches according to which one may deal with geometry. Roughly speaking, possible approaches are:

- manipulative
- intuitive
- deductive
- analytic.

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Also one may distinguish between a geometry which stresses "static" properties of geometric objects and a geometry where objects are considered in a "dynamic" setting, as they change under the effect of different types of space transformations.

3. Is there a crisis in the teaching of geometry?

During the second half of this century geometry seems to have progressively lost its former central position in mathematics teaching in most countries. The decrease has been both qualitative and quantitative. Symptoms of this decrease may be found for instance in recent national and international surveys on the mathematical knowledge of students. Often geometry is totally ignored or only a very few items concerned with geometry are included. In the latter case questions tend to be confined to some elementary "facts" about simple figures and their properties, and performance is reported to be relatively poor.

What are the main causes of this situation?

– From about 1960 to 1980 a general time pressure on traditional topics has occurred, due to the introduction of new topics in mathematics curricula (e.g. probability, statistics, computer science, discrete mathematics). At the same time the number of school hours devoted to mathematics has gone down. The "modern mathematics movement" has contributed – at least indirectly – to the decline of the role of euclidean geometry, favouring other aspects of mathematics and other points of view for its teaching (e.g. set theory, logic, abstract structures). The decline has involved in particular the role of visual aspects of geometry, both three-dimensional and two-dimensional, and all those parts which did not fit into a theory of linear spaces as, for instance, the study of conic sections and of other noteworthy curves.

– In more recent years there has been a shift back towards more traditional contents in mathematics, with a specific emphasis on problem posing and problem solving activities. However, attempts to restore classical euclidean geometry – which earlier in many parts of the world was the main subject in school geometry – have so far not been very successful. The point is that in traditional courses on euclidean geometry the material is usually presented to students as a ready-made end product of mathematical activity. Hence, in this form, it does not fit well into curricula where pupils are expected to take an active part in the development of their mathematical knowledge.

– In most countries the percentage of young people attending secondary school has increased very rapidly during the last decades. Thus the traditional way of teaching abstract geometry to a selected minority has become both more difficult and more inappropriate for the expectations of the majority of students of the new generations. At the same time, the need for more teachers has caused, on average, a decline in their university preparation, especially with respect to the more demanding parts of mathematics, in particular geometry. Since younger teachers have learned mathematics under curricula that neglected geometry, they lack a good background in this field, which in turn fosters in them the tendency to neglect the teaching of geometry to their pupils.

The situation is even more dramatic in those countries which lack a prior tradition in schooling. In some cases geometry is completely absent from their mathematics curricula.

– The gap between the conception of geometry as a research area and as a subject to be taught in schools seems to be increasing; but so far no consensus

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has been found on how to bridge this gap, nor even whether it could (or should) be bridged through an introduction of more advanced topics in school curricula at lower grades.

4. Geometry as reflected in education

In former sections, we have considered geometry mainly as a mathematical theory and have analyzed some aspects of its *teaching*. Since *learning* is unquestionably the other essential pole of any educational project, it is now appropriate to pay due attention to the main variables which may affect a coherent teaching/learning process. Consequently, several different aspects or "dimensions" (considered in their broadest meaning) must be taken into account:

– The *social dimension*, with two poles:

• The cultural pole, i.e. the construction of a common background (knowledge and language) for all the people sharing a common civilization;

• The educational pole, i.e. the development of criteria, internal to each individual, for self consistency and responsibility.

– The *cognitive dimension*, i.e. the process which, starting from reality, leads gradually to a refined perception of space.

– The *epistemologic dimension*, i.e. the ability to exploit the interplay between reality and theory through modelling (make previsions, evaluate their effects, reconsider choices). Thereby axiomatization enables one to get free from reality; this in turn may be seen as a side-step which allows further conceptualization.

– The *didactic dimension*, i.e. the relation between teaching and learning. Within this dimension several aspects deserve consideration. As an example, we list three of them:

• To make various fields interact (both within mathematics and between mathematics and other sciences).

• To make sure that the viewpoints of the teacher and the pupils are consistent in a given study. For instance, to be aware that different distance scales may involve different conceptions and processes adopted by the pupils, even though the mathematical situation is the same: in a "space of small objects", visual perception may help to make conjectures and to identify geometric properties; when dealing with the space where we are used to move around (the classroom, for instance) it is still easy to get local information, but it may be difficult to achieve an overall view; in a "large scale space" (as is the case in geography or in astronomy) symbolic representations are needed in order to analyze its properties.

• To pay due consideration to the influence of tools available in teaching/learning situations (from straightedge and compass, as well as other concrete materials, to graphic calculators, computers and specific software).

It goes without saying that all these dimensions are interrelated with each other and that they should also be related appropriately to different age levels and school types: pre-primary level, primary level, lower secondary level, upper secondary level (where differentiation into academic, technical, vocational tracks usually starts), tertiary (i.e. university) level, including teacher preparation.

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5. New technology and teaching aids for geometry

There is a long tradition of mathematicians making use of technological tools, and conversely the use of these tools has given rise to many challenging mathematical problems (e.g. straightedge and compass for geometric constructions, logarithms and mechanical instruments for numerical computations). In recent years new technology, and in particular computers, has affected dramatically all aspects of our society. Many traditional activities have become obsolete, while new professions and new challenges arise. For instance, technical drawing is no longer done by hand. Nowadays, instead, one uses commercial software, plotters and other technological devices. CAD/CAM and symbolic algebra software are becoming widely available.

Computers have also made it possible to construct "virtual realities" and to generate interactively animations or marvellous pictures (e.g. fractal images). Moreover, electronic devices can be used to achieve experiences that in everyday life are either inaccessible, or accessible only as a result of time-consuming and often tedious work.

Of course, in all these activities geometry is deeply involved, both in order to enhance the ability to use technological tools appropriately, and in order to interpret and understand the meaning of the images produced.

Computers can be used also to gain a deeper understanding of geometric structures thanks to software specifically designed for didactical purposes. Examples include the possibility of simulating traditional straightedge and compass constructions, or the possibility of moving basic elements of a configuration on the screen while keeping existing geometric relationships fixed, which may lead to a dynamic presentation of geometric objects and may favour the identification of their invariants.

Until now, school practice has been only marginally influenced by these innovations. But in the near future it is likely that at least some of these new topics will find their way into curricula. This will imply great challenges:

– How will the use of computers affect the teaching of geometry, its aims, its contents and its methods?

– Will the cultural values of classical geometry thereby be preserved, or will they evolve, and how?

– In countries where financial constraints will not allow a massive introduction of computers into schools in the near future, will it nevertheless be possible to restructure geometry curricula in order to cope with the main challenges originated by these technological devices?

6. Key issues and challenges for the future

In this section we list explicitly some of the most relevant questions which arise from the considerations outlined in the preceding sections. We believe that a clarification of these issues would contribute to a significant improvement in the teaching of geometry. Of course we do not claim that all the problems quoted below are solvable, nor that the solutions are unique and have universal validity. On the contrary, the solutions may vary according to different school levels, different school types and different cultural environments.

6.1. Aims

Why is it advisable and/or necessary to teach geometry?

Society Version

Which of the following may be considered to be the most relevant aims of the teaching of geometry?

– To describe, understand and interpret the real world and its phenomena.

– To supply an example of an axiomatic theory.

– To provide a rich and varied collection of problems and exercises for individual student activity.

– To train learners to make guesses, state conjectures, provide proofs, and find out examples and counterexamples.

– To serve as a tool for other areas of mathematics.

– To enrich the public perception of mathematics.

6.2. Contents

What should be taught?

Is it preferable to emphasize "breadth" or "depth" in the teaching of geometry? And is it possible/advisable to identify a core curriculum?

In the case of an affirmative answer to the second question above, which topics should be included in syllabi at various school levels?

In the case of a negative answer, why is it believed that teachers or local authorities should be left free to choose the geometric contents according to their personal tastes (is this point of view common to other mathematical subjects, or is it peculiar to geometry)?

Should geometry be taught as a specific, separate subject, or should it be merged in general mathematical courses?

There seems to be widespread agreement that the teaching of geometry must reflect the actual and potential needs of society. In particular, geometry of three-dimensional space should be stressed at all school levels, as well as the relationships between three-dimensional and two-dimensional geometry. How could and should the present situation (where only two-dimensional geometry is favoured) therefore be modified and improved?

In which ways can the study of linear algebra enhance the understanding of geometry? At what stage should "abstract" vector space structures be introduced? And what are the goals?

Would it be possible and advisable also to include some elements of non-euclidean geometries into curricula?

6. 3. Methods

How should we teach geometry?

Any topic taught in geometry can be located somewhere between the two extremes of an "intuitive" approach and a "formalized" or "axiomatic" approach. Should only one of these two approaches be stressed at each school level, or should there be a dialectic interplay between them, or else should there be a gradual shift from the former to the latter one, as the age of students and the school level progresses?

What is the role of axiomatics within the teaching of geometry? Should a complete set of axioms be stated from the beginning (and, if so, at what age and school level) or is it advisable to introduce axiomatics gradually, e.g. via a "local deductions" method?

Traditionally, geometry is the subject where "one proves theorems". Should "theorem proving" be confined to geometry?

Society Edition

Would we like to expose students to different levels of rigour in proofs (as age and school level progress)? Should proofs be tools for personal understanding, for convincing others, or for explaining, enlightening, verifying?

Starting from a certain school level, should every statement be proved, or should only a few theorems be selected for proof? In the latter case, should one choose these theorems because of their importance within a specific theoretical framework, or because of the degree of difficulty of their proof? And should intuitive or counterintuitive statements be privileged?

It seems that there is an international trend towards the teaching of analytic methods in increasingly earlier grades, at the expense of other (synthetic) aspects of geometry. Analytic geometry is supposed to present algebraic models for geometric situations. But, as soon as students are introduced to these new methods, they are suddenly projected into a new world of symbols and calculations in which the link between geometric situations and their algebraic models breaks down and geometric interpretations of numerical calculations are often neglected. Hence, at what age and school level should teaching of analytic geometry start? Which activities, methods and theoretical frameworks can be used in order to restore the link between the algebraic representation of space and the geometric situation it symbolizes?

How can we best improve the ability of pupils to choose the appropriate tools for solving specific geometric problems (conceptual, manipulative, tecnological)?

6.4. Books, computers, and other teaching aids

Are traditional textbooks as appropriate for teaching and learning geometry, as we would like them to be?

How do teachers and pupils actually use geometry textbooks and other teaching aids? How would we like pupils to use them?

What changes could and should be made in teaching and learning geometry in the perspective of increased access to software, videos, concrete materials and other technological devices?

What are the advantages, from the educational and geometrical point of view, that can follow from the use of such tools?

Which problems and limitations may arise from the use of such tools, and how can they be overcome?

To what extent is knowledge acquired in a computer environment transferable to other environments?

6.5. Assessment

The ways of assessment and evaluation of pupils strongly influence teaching and learning strategies. How should we set out objectives and aims, and how should we construct assessment techniques that are consistent with these objectives and aims? Are there issues of assessment which are peculiar to the teaching and learning of geometry?

How does the use of calculators, computers and specific geometric software influence examinations as regards content, organization and criteria for the evaluation of the answers of the students?

Should assessment procedures be based mainly upon written examination papers (as it seems to be customary in many countries) or else what should be the role of oral communication, of technical drawing and of work with the computer?

Society Edition

What is it exactly that should be evaluated and considered for assessment: The solution outcome? The solution process? The method of thinking? Geometric constructions?

6.6. Teacher preparation

One essential component of an efficient teaching/learning process, is good teacher preparation, as regards both disciplinary competence and educational, epistemological, technological and social aspects. Hence, what specific preparation in geometry is needed (and realistically achievable) for prospective and practicing teachers?

It is well known that teachers tend to reproduce in their profession the same models they experienced when they were students, regardless of subsequent exposure to different points of view. How is it then possible to motivate the need for changes in the perspective of teaching geometry (both from the content and from the methodological point of view)?

Which teaching supplies (books, videos, software,...) should be made available for in-service training of teachers, in order to favour a flexible and open-minded approach to the teaching of geometry?

6.7. Evaluation of long-term effects

All too often the success (or failure) of a curricular and/or methodological reform or innovation for a certain school system is evaluated on the basis only of a short period of observation of its outcomes. Moreover usually there are no comparative studies on the possible side effects of a change of content or methods. Conversely, it would be necessary to look also at what happens in the long term. For instance:

– Does a visual education from a very young age have an impact on geometric thinking at a later stage?

– How does an early introduction of analytic methods in the teaching of geometry influence the visual intuition of pupils? When these pupils become professionals, do they rely more on the intuitive or on the rational parts of the geometry teaching to which they have been exposed?

– What is the impact of an extensive use of technological tools on geometry learning?

6.8. Implementation

At ICME 5 (Adelaide, 1984) J. Kilpatrick asked a provocative question: What do we know about mathematics education in 1984 that we did not know in 1980? Recently the same question has been picked up again in the ongoing ICMI study: "What is research in mathematics education, and what are its results". As for geometry, the possibility of relying on research results would be extremely useful in order to avoid reproposing in the future paths already proved unsuccessful, and conversely in order to benefit from successful solutions. And, as for still unsettled and relevant questions, we would like research to give us useful information in order to clarify the advantages and drawbacks of possible alternatives.

Hence, a key question might be:

What do we already know from research about the teaching and learning of geometry and what would we want future research to tell us?

Survey Edition

7. Call for papers

The ICMI study **"Perspectives on the Teaching of Geometry for the 21st Century"** will consist of an invited *Study Conference* and a *Publication* to appear in the ICMI study series, based on the contributions to, and the outcomes of, the Conference.

The Conference is scheduled for September 1995 in Catania (Italy). The International Program Committee (IPC) for the study hereby invites individuals and groups to submit ideas, suggestions and contributions on major problems or issues related to this discussion document, not later than February 15, 1995.

Although participation in the conference requires an invitation from the IPC, "experts" and "newcomers" interested in contributing to and participating in the conference are encouraged to contact the chair of the IPC. Unfortunately, an invitation to attend does not imply that financial support will be provided by the organizers.

Papers, as well as suggestions concerning the content of the study conference program should be sent to

Prof. Vinicio VILLANI
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
I - 56127 PISA , ITALY
e-mail: <villani@dm.unipi.it>

The IPC members are:

Vinicio VILLANI (Chair of the IPC). Carmelo MAMMANA (Chair of the Local Organizing Committee, Dipartimento di Matematica, Viale A. Doria 6, Città Universitaria, I-95125 Catania, Italy, e-mail: <mammana@dipmat.unict.it>), Régine DOUADY (IREM, Univ. Paris VII, France), Vagn Lundsgaard HANSEN (Math. Inst., Technical Univ. of Denmark, Lyngby, Denmark), Rina HERSHKOWITZ (Dept. of Science Teaching, the Weizmann Inst. of Science, Rehovot, Israel), Joseph MALKEVITCH (Math., York College, CUNY, Jamaica, N.Y., USA), Iman OSTA (American Univ. of Beirut, Lebanon), Mogens NISS (Member ex officio, IMFUFA, Roskilde Univ., Denmark).

Socio-Edition

List of Participants to the Catania Conference

N. Aguilera (Argentina)
A. Ambrus (Hungary)
M. Bartolini Bussi (Italy)
R. Berthelot (France)
R. Bkouche (France)
P. Boero (Italy)
G. Brown (USA)
S. Casella (Uruguay)
A. Chronaki (England)
B. D'Amore (Italy)
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R. Douady (France)
R. Duval (France)
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R. Hershkowitz (Israel)
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L. Jones (England)
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Z. Lučić (Serbia)
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G. Malaty (Finland)
J. Malkevitch (USA)
C. Mammana (Italy)
M. Marchi (Italy)
M. A. Mariotti (Italy)
P. Maroscia (Italy)
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N. Pietrocola (Argentina)
R. Porcaro (Italy)
M. Rahim (Canada)
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J. Ruberu (Brunei)
H. Schumann (Germany)
F. Speranza (Italy)
R. Straesser (Germany)
D. Taimina (Lettland)
S. C. Tang (China)
D. Torri (Italy)
S. Turnau (Poland)
C. Vasco (Colombia)
V. Villani (Italy)
F. Zhang (China)

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Addresses of the Contributors

Maria G. Bartolini Bussi
Dip. di Matematica Pura ed Appl.
Università, Via Campi 213/b
41100 Modena, Italy

René Berthelot
LADIST, Université Bordeaux 1
40, rue Lamartine
33400 Talence, France

Paolo Boero
Dip. di Matematica, Università
Via Dodecaneso 35
16146 Genova, Italy

Adrien Douady
Dép. de Mathématiques
Université de Paris-Sud
91405 Orsay, France

Régine Douady
IREM - Université Paris 7, C.P. 7018
2, Pl. Jussieu
75251 Paris 05, France

Raymond Duval
Université du Littoral
IUFM du Nord Pas-de-Calais
59820 Gravelines, France

Massimo Galuzzi
Dip. di Matematica, Università
Via Saldini, 50
20133 Milano, Italy

Ghada K. Gholam
Dar Al Handasah
P.O.Box 895
Cairo, Egypt

Klaus-Dieter Graf
Freie Universität Berlin
Institut für Informatik
D-14195 Berlin, Germany

H. Brian Griffiths
University of Southampton
Faculty of Mathematical Studies
Highfield, Southampton
SO17 1BJ, England
&
Mathematics Centre
Chichester Institute of
Higher Education,
Bognor Regis, West Sussex,
PO21 1HR, England

Angel Gutiérrez
Universitat de València,
Dep. de Did. de la Matemàtica
46071 Valencia, Spain

Vagn Lundsgaard Hansen
Dept. of Mathematics,
Technical University of Denmark,
Building 303, DK-2800,
Lyngby, Denmark

Rina Hershkowitz
Dept. of Science Teaching
Weizmann Institute of Science
76100 Rehovot, Israel

Bernard R. Hodgson
Dép. de Mathém. et de Statist.
Université Laval,
Québec G1K 7P4 Canada

Celia Hoyles
Institute of Education,
University of London,
London, WC1H OAL, England

Pedro Huerta
Universitat de València,
Dep. de Did. de la Matemàtica,
46071 Valencia, Spain

Keith Jones
University of Southampton
School of Education
Highfield, Southampton,
SO17 1BJ, England

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Colette Laborde
DidaTech LSD, BP 53 University
38041 Grenoble
France

Peter Legiša
Dept. of Mathematics, University
Jadranska 19, 1000 Ljubljana
Slovenia

Richard Lehrer
NCRMSE - University of Wisconsin
Madison WI, 53706 USA

Joseph Malkevitch
Mathematics and Computing Dept.
York College (CUNY)
Jamaica, New York 11451-0001
USA

Carmelo Mammana
Dip. di Matematica, Città Univ.
Viale A. Doria 6
95125 Catania, Italy

Walter Meyer
Adelphi University
Garden City, New York 11530
USA

Michael Neubrand
Universität Flensburg
Inst. für Mathematik und ihre Didaktik
Mürwiker Str. 77
D-24943 Flensburg, Germany

Mogens Niss
IMFUFA, Roskilde University
P. O. Box 260
DK-4000 Roskilde, Denmark

Iman Osta
American University of Beirut
Dept. of Education
P. O. Box 11-0236
Beirut, Lebanon

Bernard Parzysz
IUFM de Lorraine,Université
Metz, France

John Pegg
Dept. of Curriculum Studies,
University of New England
Armidale NSW 2351, Australia

Thomas Romberg
NCRMSE – University of Wisconsin
Madison, WI, 53706 USA

M. H. Salin,
LADIST, Université Bordeaux 1
40, rue Lamartine
33400 Talence, France

Tang Sheng Chang
Shanghai High School
400 Shang Zhong Road
200231 Shanghai, P. R. China

Jerzy Tocki
Instytut Matematyki
Wyzszej Szkoly Pedagogicznej
Ul. Rejtana 16 A
35-310 Rzeszow, Poland

Stefan Turnau
Instytut Matematyki
Wyzszej Szkoly Pedagogicznej
Ul. Rejtana 16 A
35-310 Rzeszow, Poland

Carlos E. Vasco
Universidad Nacional de Colombia
Avenida 32, n. 15-31
Santafe de Bogota, 1, DC, Colombia

Vinicio Villani
Dip. di Matematica, Università,
Via Buonarroti, 2
56127 Pisa, Italy

Zhang Fusheng
Educational Commission of Shanghai
200041 Shanghai, P. R. China

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